



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: September 14, 2015

ACCEPTED: November 9, 2015

PUBLISHED: November 24, 2015

SH^c realization of minimal model CFT: triality, poset and Burge condition

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ABSTRACT: Recently an orthogonal basis of \mathcal{W}_N -algebra (AFLT basis) labeled by N -tuple Young diagrams was found in the context of 4D/2D duality. Recursion relations among the basis are summarized in the form of an algebra SH^c which is universal for any N . We show that it has an \mathfrak{S}_3 automorphism which is referred to as triality. We study the level-rank duality between minimal models, which is a special example of the automorphism. It is shown that the nonvanishing states in both systems are described by N or M Young diagrams with the rows of boxes appropriately shuffled. The reshuffling of rows implies there exists partial ordering of the set which labels them. For the simplest example, one can compute the partition functions for the partially ordered set (poset) explicitly, which reproduces the Rogers-Ramanujan identities. We also study the description of minimal models by SH^c. Simple analysis reproduces some known properties of minimal models, the structure of singular vectors and the N -Burge condition in the Hilbert space.

KEYWORDS: Conformal and W Symmetry, Field Theories in Lower Dimensions

ARXIV EPRINT: [1509.01000](https://arxiv.org/abs/1509.01000)

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1 Introduction

Many years ago [1], \mathcal{W} -algebra was formulated as a nonlinear generalization of the two dimensional conformal field theory and has been playing significant roles in many branches of physics, such as string theory, quantum gravity, the statistical mechanics, and the exactly solvable systems. It was defined as an extended conformal symmetry with higher spin currents. In the case of \mathcal{W}_N -algebra, the generators of the symmetry consist of spin $2, 3, \dots, N$ currents. The commutation relations among them are in general nonlinear and the explicit form of the algebra is known only for simple cases such as \mathcal{W}_3 . For the general cases, while the algebra itself is not given explicitly, the representation was derived through the realization with free bosons [2].

Recently, in the efforts to prove the 4D/2D correspondence [3] a new realization of A-type \mathcal{W} -algebra was constructed [4–6]. It is based on the orthogonal AFLT basis (Alba-Fateev-Litvinov-Tarnopolskiy basis) labeled by an array of Young diagrams [7, 8]. The algebra has a rather lengthy name (central extension of spherical degenerate double affine Hecke algebra) but was abbreviated to SH^c in [4], which will be used in this paper. The generators of the algebra, $D_{r,l}$, are labeled by two integers,¹ $r \in \mathbb{Z}$ and $l \in \mathbb{N}$. The essential part of the algebra is written explicitly in terms of $D_{\pm 1,l}$ and $D_{0,l}$.

The rank N representation of SH^c is spanned by N -tuple Young diagrams and can be realized in terms of N free bosons. The explicit forms of the generators are given for $D_{r,0}$ and $D_{r,1}$. The generators $D_{r,0}$ are identified with free $\text{U}(1)$ current and some combinations of $D_{r,0}$ and $D_{r,1}$ are identified as the Virasoro generators in \mathcal{W}_N -algebra. The other \mathcal{W}_N currents are given in terms of higher generators (in less manifest fashion) in [4, 9]. In this way, the generators of two algebras can be identified through the bosonic oscillators. We note that N is arbitrary while the algebra is the same. In this sense, SH^c may be regarded as a universal \mathcal{W} -symmetry which contains representations of all \mathcal{W}_N -algebra.

The purpose of the paper is to find more direct link between the representations of \mathcal{W}_N -algebra and SH^c . Our first focus is the explicit realization of the level-rank duality [10, 11] in the SH^c module. It is a duality between minimal models in \mathcal{W}_N - and \mathcal{W}_M -algebras with $N \neq M$. This is somehow puzzling since we need to find a direct correspondence between two Hilbert spaces spanned by different number of Young diagrams. Another motivation to study the duality is that it would be related to the so-called triality symmetry of $\mathcal{W}_\infty[\mu]$ [12, 13]. In our notation, it is realized by a discrete non-abelian automorphism \mathfrak{S}_3 through the transformation of a parameter β . It is generated by an obvious transformation $\beta \leftrightarrow 1/\beta$ and less obvious one $\beta \leftrightarrow \frac{\beta}{\beta-1}$. We show that this automorphism holds in SH^c in general. The level-rank duality is interesting since they are realized among the minimal model CFTs where we need to impose β to take a specific form.

We note that the AFLT basis is diagonal with respect to infinite commuting operators in SH^c . The eigenvalues are the power sum of numbers assigned to each box of the Young diagrams. By analyzing the integer assigned to each box, we find that the two Hilbert spaces are related by reshuffling rows contained in each Young diagrams (see figure 1). To be specific, we show that there is a set of numbers X which labels the rows of Young

¹We use the convention $0 \in \mathbb{N}$.

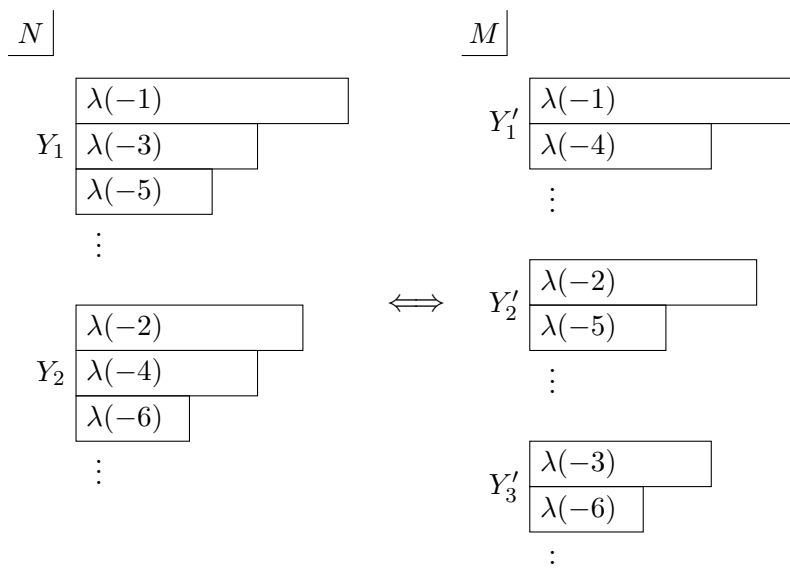


Figure 1. An example of reshuffling rows as a realization of the level-rank duality ($N = 2, M = 3$). We take the set of labels $X = \{-1, -2, \dots\}$ which represents the rows.

diagrams. The set X is determined by the representation of \mathcal{W}_N (\mathcal{W}_M). For each $x \in X$, we assign a positive number $\lambda(x)$ which satisfies both $\lambda(x) \geq \lambda(x - N)$ and $\lambda(x) \geq \lambda(x - M)$. We show that the N -(resp. M -) tuple Young diagrams \vec{Y} (resp. \vec{Y}') in \mathcal{W}_N (resp. \mathcal{W}_M) is described by such a partition.

The counting of states with such property was studied in the literature [14] which is referred as the partition function of poset (partially ordered set). In the simplest case $(N, M) = (2, 3)$, one can solve the combinatorics completely to obtain the formula for the partition function. It turns out that the summation takes the same form as Rogers-Ramanujan formulae and agree with the known character formula (for example [11]) of the minimal models of the \mathcal{W} -algebra.

The Hilbert space for SH^c should coincide with that of \mathcal{W}_N module. To confirm it, we also study the description of the Hilbert space for general minimal models. In particular, we show that the singular vectors in the Hilbert space have a simple graphical interpretation and they agree with those for \mathcal{W}_N module. We also demonstrate explicitly that the condition which characterizes the Hilbert space of the minimal models, the N -Burge condition which was proposed recently [15], by showing that any states which violate the condition are not produced by the operators in SH^c .

This paper is organized as follows. In section 2, we review some basic aspects of minimal models of \mathcal{W}_N -algebra, the level-rank duality and the triality. We explain the correspondence of the conformal dimensions in some detail to prepare the notation in the following sections. A brief summary of the SH^c is given in section 3. We show how the triality is realized in SH^c . It is an exact automorphism of the algebra. In section 4, we describe the Hilbert space of the minimal models by SH^c . We show that the null states appearing in the Hilbert space agree with those for \mathcal{W}_N module. In section 5, we prove that

the central charges of SH^c agree for a pair of minimal models with the level-rank duality. While the triality is an exact automorphism of SH^c in general, this example is special since it is realized through the tuning of finite parameters. In section 6, we demonstrate the duality at the state-to-state level. The correspondence between the partition of the poset and Young diagrams mentioned above is explained in detail. In particular we prove the partition for poset gives the Hilbert space correctly. In section 7, we explain how to calculate the partition function for the poset according to [14]. For the simplest case (2, 3) the computation can be performed exactly and equality with the known partition function gives the Rogers-Ramanujan identity. In section 8, we come back to the general minimal models. We study the action of generators of SH^c and derive the N -Burge condition.

2 Brief review of level-rank duality in \mathcal{W} -algebras

In this section, we recall some basic facts about minimal models and the level-rank duality that will be important later. Especially a way to identify the dual primary fields for the level-rank duality pair is given at the end of this section in detail and we will see a similar duality in SH^c in later sections.

2.1 Minimal models of \mathcal{W}_N -algebra

A minimal model of the \mathcal{W}_N -algebra [2, 16] is characterized by a central charge,

$$c = (N - 1) \left(1 - \frac{(p - q)^2}{pq} N(N + 1) \right), \quad (2.1)$$

which is parametrized by a pair of coprime positive integers (p, q) with $p, q \geq N$. The highest weights of primary fields are limited to

$$\Delta(n_1, \dots, n_{N-1}; n'_1, \dots, n'_{N-1}) = \frac{12 \left(\sum_{i=1}^{N-1} (pn_i - qn'_i) \vec{\omega}_i \right)^2 - N(N^2 - 1)(p - q)^2}{24pq}, \quad (2.2)$$

where two sequences $(n_i)_{i=1}^{N-1}$ and $(n'_i)_{i=1}^{N-1}$ consist of positive integers subjected to the following inequalities,

$$\sum_{i=1}^{N-1} n_i \leq q - 1, \quad \sum_{i=1}^{N-1} n'_i \leq p - 1, \quad (2.3)$$

and $\{\vec{\omega}_i\}_{i=1}^{N-1}$ are the fundamental weights of $\mathfrak{su}(N)$ satisfying,

$$\vec{\omega}_i \cdot \vec{\omega}_j = \frac{i(N - j)}{N}, \quad i \leq j. \quad (2.4)$$

The minimal model is realized by the coset construction in terms of $\text{SU}(N)$ current algebra,

$$\mathcal{W}_{N,k} \equiv \frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}, \quad (2.5)$$

with the level k given as

$$k = \frac{p}{q - p} - N. \quad (2.6)$$

This model is unitary when k is a non-negative integer.

The Hilbert space associated with a highest weight $\Delta((n_i), (n'_i))$ is completely degenerate in a sense that there are N singular vectors² appearing at the levels $n_i n'_i$ ($1 \leq i \leq N$), where $n_N \equiv q - \sum_{i=1}^{N-1} n_i$ and $n'_N \equiv p - \sum_{i=1}^{N-1} n'_i$. Note that the inequalities (2.3) ensure that n_N and n'_N are positive integers and we have

$$\sum_{i=1}^N n_i = q, \quad \sum_{i=1}^N n'_i = p. \quad (2.7)$$

2.2 Level-rank duality

The level-rank duality is an extra symmetry which interpolates a \mathcal{W}_N -algebra with another \mathcal{W}_M -algebra [10, 11] when the parameters of the minimal model p, q take special values. It is written as a correspondence between coset models:

$$\mathcal{W}_{N,k} = \frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \sim \frac{\mathfrak{su}(M)_l \oplus \mathfrak{su}(M)_1}{\mathfrak{su}(M)_{l+1}} = \mathcal{W}_{M,l}, \quad (2.8)$$

where N and M are coprime and the levels k, l are given as

$$k = \frac{N}{M} - N, \quad l = \frac{M}{N} - M. \quad (2.9)$$

Both models have the same central charge:

$$\begin{aligned} (N-1) \left(1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right) &= (M-1) \left(1 - \frac{M(M+1)}{(M+l)(M+l+1)} \right) \\ &= -\frac{(N-1)(M-1)(N+M+NM)}{N+M}. \end{aligned} \quad (2.10)$$

We can see from (2.6) that the model $\mathcal{W}_{N,k}$ corresponds to the one with $p = N$ and $q = N + M$. Since $k \notin \mathbb{N}$, the model is not unitary.

The inequalities (2.3) imply that $n'_i = 1$ for all $1 \leq i \leq N-1$. We introduce the notation $\tilde{n}_i \equiv n_i - 1 \geq 0$, which are then constrained by $\sum_{i=1}^{N-1} \tilde{n}_i \leq M$, and denote highest weights by

$$\Delta(\tilde{n}) \equiv \Delta(n_1, \dots, n_{N-1}; 1, \dots, 1). \quad (2.11)$$

Note that the conformal dimension $\Delta(\tilde{n})$ is simplified to

$$\Delta(\tilde{n}) = \frac{1}{2(N+M)} \left(\sum_{i=1}^{N-1} i(N-i)(\tilde{n}_i^2 - M\tilde{n}_i) + 2 \sum_{i < j}^{N-1} i(N-j)\tilde{n}_i\tilde{n}_j \right), \quad (2.12)$$

by using

$$\vec{\rho} \equiv \sum_i \vec{\omega}_i, \quad \vec{\rho} \cdot \vec{\rho} = \frac{1}{12} N(N^2 - 1), \quad \vec{\rho} \cdot \vec{\omega}_i = \frac{i(N-i)}{2}. \quad (2.13)$$

²The existence of the N -th singular vector comes from \mathbb{Z}_N symmetry of \mathcal{W}_N algebra, which was written in [2].

2.3 Triality

It is sometimes useful to introduce a parametrization for the central charge as

$$c = (N-1)(1-Q^2N(N+1)), \quad Q = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}. \quad (2.14)$$

For minimal models, we set $\beta = q/p$. This formula has an obvious symmetry (duality) in the form:

$$\sigma_1 : \beta \mapsto \frac{1}{\beta}, \quad N \mapsto N. \quad (2.15)$$

The level-rank duality symmetry is realized for a specific choice:

$$\beta = \frac{N+M}{N}. \quad (2.16)$$

We will see later that it is realized by the following transformation,

$$\sigma_2 : \beta \mapsto \frac{\beta}{\beta-1}, \quad N \mapsto M = (\beta-1)N. \quad (2.17)$$

The two transformations $\sigma_{1,2}$ do not commute with each other and generate the symmetric group \mathfrak{S}_3 . It is isomorphic to the modular group of four points on two sphere. We note that such a symmetry exists only for specific choices of the parameter β .

Such automorphism of algebra is referred as “triality” in the literature on $\mathcal{W}_\infty[\mu]$ algebra [12, 13]. It is the automorphism of the universal \mathcal{W} -symmetry but holds for the general choice of parameters. In the following, we show that a similar automorphism exists for SH^c which is also the universal \mathcal{W} symmetry in a sense that its rank N representation agrees with the representation of \mathcal{W}_N algebra for arbitrary N . Some difference exists, however, since we do not take the large N limit.

2.4 Correspondence of primary fields

In this subsection, we explain the duality of primary fields (their number and conformal dimensions) of the minimal models with the level-rank duality with (N, M) in details. Although they are well-known, we need to introduce a graphical representation which will be useful in later sections.

We start from the number of primary states. It is given by

$$\frac{(N+M-1)!}{N!M!}, \quad (2.18)$$

which is symmetric between N and M . It may be explained as follows. As in the previous subsection we introduce \tilde{n}_N such that $\sum_{i=1}^N \tilde{n}_i = M$. By using them we introduce the number of occupations $\{x_i\}_{i=1}^N$ in the cyclic group $\mathbb{Z}_{N+M} = \{1, 2, \dots, N+M\}$ such that $x_1 = N+M$;

$$x_1 = N+M > x_2 > \dots > x_N \geq 1. \quad (2.19)$$

\tilde{n}_i can be derived from such set by

$$\begin{aligned} x_i - x_{i+1} &= n_i = \tilde{n}_i + 1, \quad (1 \leq i \leq N-1), \\ x_N &= n_N = \tilde{n}_N + 1. \end{aligned} \quad (2.20)$$

The allowed choices of x_i are counted as $\frac{(N+M-1)!}{(N-1)!M!}$. We will show just after (2.21) that there is an overcounting associated with the rotational symmetry \mathbb{Z}_N ($\tilde{n}_i \mapsto \tilde{n}_{i+1}$ where i is defined as mod N). We need to divide it by N and it gives (2.18).

We write a diagram of a disk divided by $N + M$ parts to represent the \mathbb{Z}_{N+M} . In the diagram we shadow the parts associated with the set $\{x_i\}$ (see figure 2). \tilde{n}_i is interpreted as the size of the unoccupied parts between x_i and x_{i+1} with $x_{N+1} = x_1$. We reorganize the set $\{x_i\}_{i=1}^N$ by identifying the adjacent numbers (in the sense of \mathbb{Z}_{N+M}) as a block. For instance, we consider the case $\{x_i\} = \{8, 5, 4\}$ as figure 2. The set is divided into two blocks $\{8\}$ and $\{5, 4\}$ and $\tilde{n} = (2, 0, 3)$. The number of blocks is equal to that of nonvanishing \tilde{n} since $\tilde{n}_i = 0$ implies x_i and x_{i+1} are adjacent.

We denote the number of blocks by $L \leq \min(N, M)$ in the following. Sometimes we relax the constraint that $N + M$ must be contained in $\{x_i\}_{i=1}^N$. We get an additional degree of freedom to shift all x_i 's with a constant number in \mathbb{Z}_{N+M} . The number of blocks remains the same under such shift.

We define the dual set of numbers $\{y_i\}_{i=1}^M$ as the unoccupied set in \mathbb{Z}_{N+M} . Obviously $N + M$ is not an element in this dual set and then we perform a shift in \mathbb{Z}_{N+M} to recover the constraint back. There are M ways of the shifts which meet the constraint, but they should be identified in the sense of giving rise to the same sequence $(m_i \equiv y_i - y_{i+1})_{i=1}^M \subset \mathbb{Z}_{N+M}$ up to \mathbb{Z}_M . Again we introduce the notation $\tilde{m}_i \equiv m_i - 1$. \tilde{m}_i is interpreted as the number of occupied numbers between two neighboring unoccupied numbers. The number of nonvanishing \tilde{m} is interpreted as the number of unoccupied blocks and identical to L defined above. We note here that each nonvanishing \tilde{m}_i expresses the size of an occupied block, which is equal to $(\# \text{ of vanishing } \tilde{n}_j\text{'s}) + 1$ appearing in the block.

To illustrate the situation, we take the example $\{x_i\} = \{8, 5, 4\}$ in figure 2 again. As the dual set, one may choose $y = (8, 6, 5, 2, 1)$ and $\tilde{m} = (1, 0, 2, 0, 0)$ (see figure 3) after a shift. The number of nonvanishing \tilde{n} 's is $L = 2$.

With this preparation, let us show the conformal dimensions of these two dual pictures are manifestly identical. From the formula (2.12),

$$\begin{aligned}
2(N + M)\Delta(\tilde{n}) &= - \sum_{i=1}^{N-1} i(N - i)\tilde{n}_i \sum_{j \neq i}^N \tilde{n}_j + 2 \sum_{i < j}^{N-1} i(N - j)\tilde{n}_i\tilde{n}_j \\
&= \sum_{i < j}^N (-i(N - i) - j(N - j) + 2i(N - j))\tilde{n}_i\tilde{n}_j \\
&= \sum_{i < j}^N (i - j)(N + i - j)\tilde{n}_i\tilde{n}_j \\
&= -\frac{1}{2} \sum_{i=1}^N \sum_{d=1}^{N-1} d(N - d)\tilde{n}_i\tilde{n}_{i+d}.
\end{aligned} \tag{2.21}$$

From the third line to the last line, we implicitly impose a periodic identification $\tilde{n}_{i+N} = \tilde{n}_i$ for labels with \tilde{n} . Here we can see that there exists a \mathbb{Z}_N symmetry as we announced before: $\Delta(\tilde{n})$ is invariant under $\tilde{n}_i \mapsto \tilde{n}_{i+1}$.

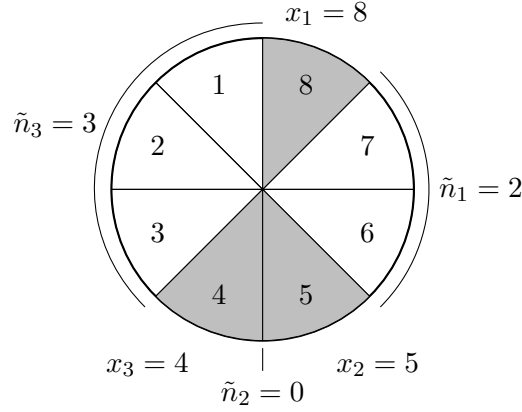


Figure 2. The interpretation of \tilde{n} .

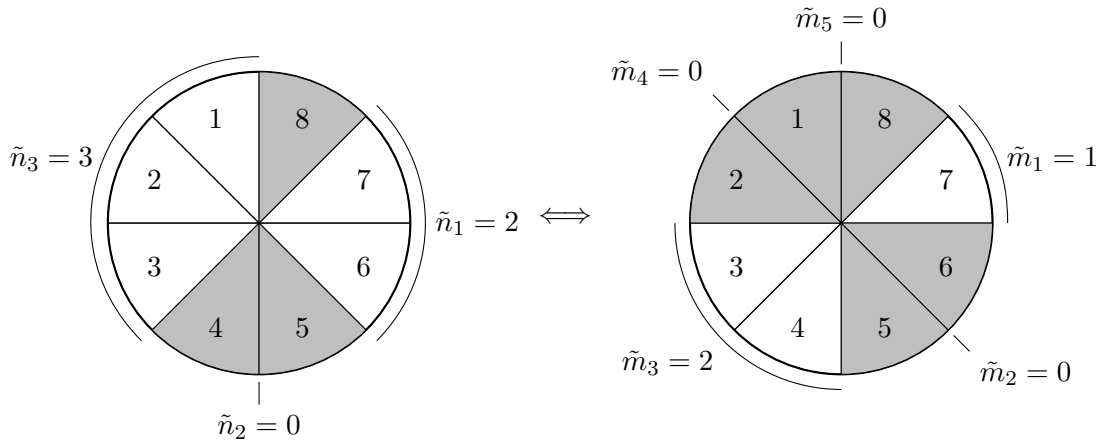


Figure 3. A sketch shows how to produce a dual set.

Let $(\tilde{n}_{\tilde{i}})_{\tilde{i}=1}^L$ denote the subsequence of all nonzero \tilde{n} 's. The nonzero contribution in the above sum comes only from two $\tilde{n}_{\tilde{i}}$'s. We can see from the interpretation of non-vanishing \tilde{m}_i 's noted above that, for such \tilde{n}_i 's, the corresponding d and $N - d$ express the sizes of occupied blocks in the two arcs between the two unoccupied blocks, respectively, and then can be expressed as summations of some nonvanishing \tilde{m}_i 's. In the example of figure 3, the nonvanishing \tilde{n}_i is \tilde{n}_1 and \tilde{n}_3 . The distance parameters are $d = 2 = \tilde{m}_3 =: \tilde{m}_1$ and $N - d = 1 = \tilde{m}_1 =: \tilde{m}_2$ (see figure 4). Now we denote the subsequence of all nonzero \tilde{m} 's by $(\tilde{m}_{\alpha})_{\alpha=1}^L$. We allocate \tilde{m}_i 's suitably so that \tilde{m}_i equals the size of the occupied block in the (clockwise) arc from the unoccupied block labeled by \tilde{n}_i to the one by \tilde{n}_{i+1} . Thus we have

$$2(N + M)\Delta(\tilde{n}) = - \sum_{\tilde{i} < \tilde{j}} \sum_{\tilde{i} \leq \beta < \tilde{j}} \sum_{\tilde{j} \leq \gamma < \tilde{i} + L} \tilde{m}_{\beta} \tilde{m}_{\gamma} \tilde{n}_{\tilde{i}} \tilde{n}_{\tilde{j}} = - \sum_{\beta < \gamma} \sum_{\beta < \tilde{j} \leq \gamma} \sum_{\gamma < \tilde{i} \leq \beta + L} \tilde{n}_{\tilde{i}} \tilde{n}_{\tilde{j}} \tilde{m}_{\beta} \tilde{m}_{\gamma}. \quad (2.22)$$

The last expression is manifestly dual by the exchange of \tilde{n} and \tilde{m} as $\tilde{m}_{\tilde{i}-1} \mapsto \tilde{n}_{\tilde{i}}$, $\tilde{n}_{\tilde{i}} \mapsto \tilde{m}_{\tilde{i}}$. Therefore, the conformal dimension is invariant under the operation $(n_i)_{i=1}^N \mapsto (m_i)_{i=1}^M$ which represents the duality transformation σ_2 .

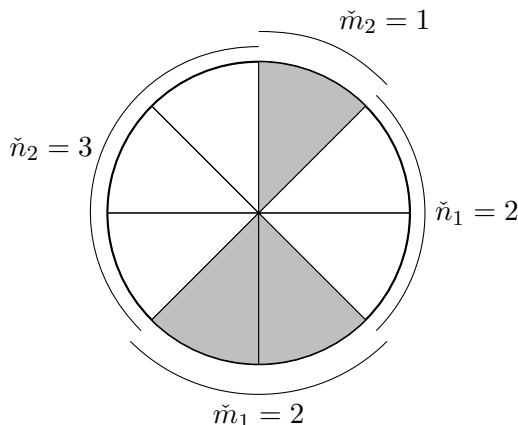


Figure 4. The way we fix the allocation of \tilde{m}_i 's.

Example: Lee-Yang edge singularity, $(N, M) = (2, 3)$. The simplest example with the duality is given when $(N, M) = (2, 3)$. It will be used repeatedly as the example to illuminate the computation. We have duality between Virasoro ($N = 2$) and \mathcal{W}_3 -algebra ($M = 3$) with the central charge $c = -\frac{22}{5}$. There are two primary fields. The diagrams like figure 3 which may be drawn in $N = 2$ (resp. $M = 3$) side are (1) $x = (5, 4)$ (resp. $y = (5, 4, 3)$) and (2) $x = (5, 3)$ (resp. $y = (5, 3, 2)$). Corresponding \tilde{n} (resp. \tilde{m}) are $\tilde{n} = (0, 3)$ (resp. $\tilde{m} = (0, 0, 2)$) for (1) and $\tilde{n} = (1, 2)$ (resp. $\tilde{m} = (1, 0, 1)$) for (2). Their conformal dimensions are derived from (2.22) as

$$\Delta(0, 3) = \Delta(0, 0, 2) = 0, \quad \Delta(1, 2) = \Delta(0, 1, 1) = -\frac{1}{5}. \quad (2.23)$$

3 Brief summary of SH^c

The purpose of the paper is to study the level-rank duality and the characterization of the minimal models in terms of the orthonormal basis labeled by multiple Young diagrams and the algebra SH^c which operates on them. The algebra was defined in [4, 5] and some applications to AGT correspondence were studied in [6, 9]. In this section, we summarize some aspects of the algebra and the representation which will be used in the following sections.

3.1 The algebra SH^c

The algebra SH^c consists of infinite number of generators $D_{r,l}$ with $r \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$. We refer to the first index r as degree and to the second index l as order. The detailed relations among the generators were explored in [5]. Here we give the part which will be relevant for our analysis:³

$$\begin{aligned} [D_{0,l}, D_{1,k}] &= D_{1,l+k-1}, & l &\geq 1, \\ [D_{0,l}, D_{-1,k}] &= -D_{-1,l+k-1}, & l &\geq 1, \\ [D_{-1,k}, D_{1,l}] &= E_{k+l}, & l, k &\geq 1, \\ [D_{0,l}, D_{0,k}] &= 0, & k, l &\geq 0, \end{aligned} \quad (3.1)$$

³A short account on the additional constraints is given in appendix B.

where E_l is a nonlinear combination of $D_{0,k}$ defined in the form of a generating function,

$$\begin{aligned} E(\zeta) &= 1 + (1 - \beta) \sum_{l \geq 0} E_l \zeta^{l+1} = \mathcal{C}(\zeta) \mathcal{D}(\zeta), \\ \mathcal{C}(\zeta) &= \exp \left(\sum_{l \geq 0} (-1)^{l+1} c_l \pi_l(\zeta) \right), \\ \mathcal{D}(\zeta) &= \exp \left(\sum_{l \geq 0} D_{0,l+1} \omega_l(\zeta) \right), \end{aligned} \quad (3.2)$$

with

$$\begin{aligned} \pi_l(\zeta) &= \zeta^l G_l (1 + (1 - \beta)\zeta), \\ \omega_l(\zeta) &= \sum_{q=1, -\beta, \beta-1} \zeta^l (G_l(1 - q\zeta) - G_l(1 + q\zeta)). \end{aligned} \quad (3.3)$$

The function $G_l(\zeta)$ is defined by the following generating functional,

$$\sum_{\ell=0}^{\infty} G_\ell(\zeta) x^\ell = -\log(\zeta) + \sum_{\ell=1}^{\infty} x^\ell \frac{\zeta^{-\ell} - 1}{\ell} = \log \left(\frac{1-x}{\zeta-x} \right). \quad (3.4)$$

The other generators $D_{r,l}$ and the algebra among them are defined recursively through these commutation relations repeatedly [4].

The generators E_l are in general nonlinear combinations of $D_{0,k}$ and their commutation relation becomes nonlinear. The parameters $(c_l)_{l \geq 0}$ in the $\mathcal{C}(\zeta)$ are central charges. They are arbitrary in general.

3.2 Triality in SH^c

We note here that there exists an \mathfrak{S}_3 automorphism of the algebra SH^c which are related to the triality in subsection 2.3. We repeat to use the same notation for two generators σ_1, σ_2 . We assume that $\beta \neq 0, 1, \infty$ in the following.

The first transformation, $\sigma_1 : \beta \mapsto \beta' = 1/\beta$, keeps the algebra invariant if we rescale the parameters and the generators as follows. We first rescale

$$\zeta' = -\beta\zeta, \quad c'_\ell = (-\beta)^{-\ell} c_\ell, \quad D'_{0,l+1} = (-\beta)^{-l} D_{0,l+1}, \quad (3.5)$$

which make $\mathcal{C}(\zeta)$ and $\mathcal{D}(\zeta)$ invariant. From the definition (3.2), the generator E_l should be rescaled as, $E'_\ell = (-\beta)^{-\ell} E_\ell$. The other parts of the algebra (3.1) are kept invariant by the replacements, $D'_{\pm 1,k} = \gamma^{\pm 1} (-\beta)^{-k} D_{\pm 1,k}$ with an arbitrary nonvanishing parameter γ . The other generators are rescaled following the commutation relations.

For the second transformation, $\sigma_2 : \beta \mapsto \frac{\beta}{\beta-1}$, $\mathcal{D}(\zeta)$ is invariant if we rescale,

$$\zeta' = (\beta-1)\zeta, \quad D'_{0,l+1} = (\beta-1)^{-l} D_{0,l+1}. \quad (3.6)$$

We have to tune the central charges by imposing $\mathcal{C}(\zeta) = \mathcal{C}'(\zeta')$ which is decomposed into an infinite number of constraints by comparing the coefficients of Taylor expansion with

respect to ζ . After taking the logarithm, one obtains,

$$\begin{aligned} \log(\mathcal{C}'(\zeta')) &= c'_0 \log(1 - \zeta) + \sum_{l=1}^{\infty} (-1)^{l+1} c'_l (\beta - 1)^l \zeta^l \frac{(1 - \zeta)^{-l} - 1}{l} \\ &= -c'_0 \zeta + \sum_{l=1}^{\infty} \zeta^{l+1} \left((1 - \beta)^l c'_l + (\text{terms written by } c'_0, \dots, c'_{l-1}) \right) \end{aligned} \quad (3.7)$$

$$\begin{aligned} \log(\mathcal{C}(\zeta)) &= c_0 \log(1 + (1 - \beta)\zeta) + \sum_{l=1}^{\infty} (-1)^{l+1} c_l \zeta^l \frac{(1 + (1 - \beta)\zeta)^{-l} - 1}{l} \\ &= (1 - \beta)c_0 \zeta + (1 - \beta) \sum_{l=1}^{\infty} \zeta^{l+1} \left((-1)^l c_l + (\text{terms written by } c_0, \dots, c_{l-1}) \right). \end{aligned} \quad (3.8)$$

Comparing the coefficient of ζ^l , one can determine c'_l through c_0, \dots, c_l as $c'_l = (\beta - 1)^{-l+1} c_l + (\text{terms written by } c_0, \dots, c_{l-1})$. Explicit forms of the first three terms are written as,

$$c'_0 = (\beta - 1)c_0, \quad c'_1 = c_1 + (1 - \beta/2)c_0, \quad c'_2 = \frac{-(6 - 7\beta + 2\beta^2)c_0 + 6(2 - \beta)c_1 + 6c_2}{6(\beta - 1)}. \quad (3.9)$$

Finally the other part of the algebra is kept invariant if

$$E'_\ell = (\beta - 1)^{1-\ell} E_\ell, \quad D'_{\pm 1, \ell} = \gamma^{\pm 1} (\beta - 1)^{1/2-\ell} D_{\pm 1, \ell}. \quad (3.10)$$

Again γ is an arbitrary nonvanishing number.

In the rank N representation which will be discussed in the next subsection, the basis of the representation space is labeled by the N -tuple Young diagrams $\vec{Y} = (Y_1, \dots, Y_N)$. The first transformation σ_1 is realized by mapping \vec{Y} into their transpose $\vec{Y}' = (Y'_1, \dots, Y'_N)$. On the other hand it is in general impossible to express the second transformation in terms of the finite rank representations since the number of the parameters in the model is finite. For specific choices of parameters that correspond to the level-rank duality, however, $\mathcal{C}(\zeta)$ is kept invariant in a nontrivial way and the correspondence between \vec{Y} becomes less trivial as we will see.

3.3 Rank N representation of SH^c

There is a well-studied representation where the basis of the Hilbert space is labeled by N -tuple Young diagrams which is referred as the rank N representation of SH^c . The representation has parameters $\vec{a} = (a_q)_{q=1}^N \in \mathbb{C}^N$ with which the central charges are written in the form,

$$c_l = \sum_{q=1}^N (a_q - \xi)^l, \quad \xi \equiv 1 - \beta. \quad (3.11)$$

With this choice for the central charges, the central charge $\mathcal{C}(\zeta)$ is simplified as [6],

$$\mathcal{C}(\zeta) = \mathcal{C}_N(\zeta, \vec{a}) \equiv \prod_{q=1}^N T(\zeta, a_q), \quad T(\zeta, a) \equiv \frac{1 + \zeta a}{1 + \zeta(a - \xi)}. \quad (3.12)$$

The algebra SH^c is realized on the vector space spanned by the orthogonal basis $|\vec{a}, \vec{Y}\rangle$, where $\vec{Y} = (Y_q)_{q=1}^N$ is an N -tuple Young diagrams. Strictly speaking \vec{a} is the parameter of the algebra but we include it to the state to specify the parameter of algebra which we use. The highest weight state is given when all the Young diagrams are trivial $\vec{Y} = \vec{\emptyset}$. It satisfies the highest weight state conditions,

$$D_{-1,l}|\vec{a}, \vec{\emptyset}\rangle = 0, \quad l = 0, 1, 2, \dots \quad (3.13)$$

with $D_{0,l}|\vec{a}, \vec{\emptyset}\rangle = 0$. For $\vec{Y} \neq \vec{\emptyset}$, they describe the excited states. The basis is simultaneous eigenvectors for the infinite number of generators $D_{0,l}$ ($l \geq 0$).⁴ The other operators $D_{\pm 1,l}$ are defined to change the number of boxes of the Young diagrams by ± 1 in all possible ways with appropriate coefficients. The actions of $D_{\pm 1,l}, D_{0,l}$ on the basis are written in a closed form [4],⁵

$$D_{-1,l}|\vec{a}, \vec{Y}\rangle = (-1)^l \sum_{q=1}^N \sum_{t=1}^{f_q} (a_q + B_t(Y_q))^l \Lambda_q^{(t,-)}(\vec{Y}) |\vec{a}, \vec{Y}^{(t,-),q}\rangle, \quad (3.14)$$

$$D_{1,l}|\vec{a}, \vec{Y}\rangle = (-1)^l \sum_{q=1}^N \sum_{t=1}^{f_q+1} (a_q + A_t(Y_q))^l \Lambda_q^{(t,+)}(\vec{Y}) |\vec{a}, \vec{Y}^{(t,+),q}\rangle, \quad (3.15)$$

$$D_{0,l+1}|\vec{a}, \vec{Y}\rangle = (-1)^l \sum_{q=1}^N \sum_{\mu \in Y_q} (a_q + c(\mu))^l |\vec{a}, \vec{Y}\rangle, \quad (3.16)$$

where $c(\mu) = \beta i - j$ for $\mu = (i, j)$ describes the coordinates ($i \geq 1$ represents the horizontal and $j \geq 1$ the vertical location, $(1, 1)$ represents the box at the left-upper corner) of the Young diagram. The symbols $A_k(Y) = \beta r_{k-1} - s_k - \xi$ and $B_k(Y) = \beta r_k - s_k$ (with $r_0 = s_{f+1} = 0$) parameterize the rectangle decomposition of Young diagram, (see figure 5). The $\vec{Y}^{(t,+),q}$ (resp. the $\vec{Y}^{(t,-),q}$) is the N -tuple Young diagrams obtained by adding a box at $(r_{t-1} + 1, s_t + 1)$ to (resp. subtracting a box at (r_t, s_t) from) Y_q of \vec{Y} , and the coefficients Λ are defined as,

$$\begin{aligned} \Lambda_p^{(k,+)}(\vec{a}, \vec{Y}) &= \left(\prod_{q=1}^N \left(\prod_{\ell=1}^{f_q} \frac{a_p - a_q + A_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + A_k(Y_p) - B_\ell(Y_q)} \prod_{\ell=1}^{f_q+1} \frac{a_p - a_q + A_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + A_k(Y_p) - A_\ell(Y_q)} \right) \right)^{1/2}, \\ \Lambda_p^{(k,-)}(\vec{a}, \vec{Y}) &= \left(\prod_{q=1}^N \left(\prod_{\ell=1}^{f_q+1} \frac{a_p - a_q + B_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + B_k(Y_p) - A_\ell(Y_q)} \prod_{\ell=1}^{f_q} \frac{a_p - a_q + B_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + B_k(Y_p) - B_\ell(Y_q)} \right) \right)^{1/2}, \end{aligned} \quad (3.17)$$

where the prime in the product symbol (\prod') represents that $(\ell, q) = (k, p)$ is excluded in the product. While they are complicated, they play the fundamental role in the analysis in this paper.

⁴For $N = 1$ case, the generators $D_{0,l}$ can be identified with the infinite number of commuting operators of Calogero-Sutherland system and the basis is identified with the (normalized) Jack polynomial J_Y . For the generic N , it is identified as a generalization. While there seems to exist no explicit proof, the basis $|\vec{a}, \vec{Y}\rangle$ may be identified as the (normalized) basis which was introduced by [7, 8].

⁵This is a generalization of Pieri rule [17] which defines the recursion relation among Jack polynomial. We use the convention of [6] where the normalized basis was used.

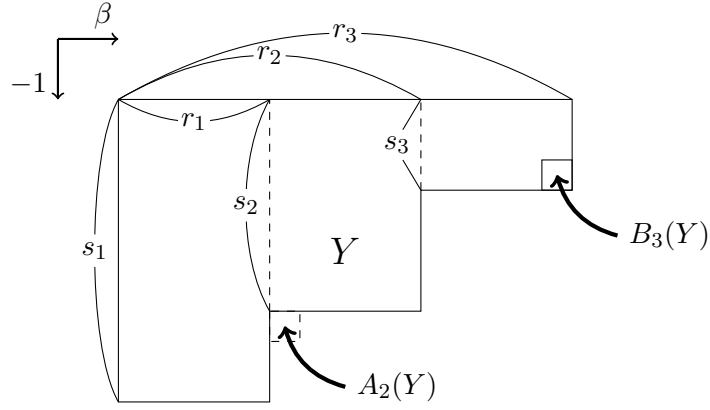


Figure 5. Rectangle decomposition of a Young diagram Y .

As the action of $D_{0,l}$ on the basis $|\vec{a}, \vec{Y}\rangle$ is diagonal, so is the action of E_l . A non-trivial part is the evaluation of the eigenvalue of $\mathcal{D}(\zeta)$ which turns out to be written in a compact form,

$$\begin{aligned} \mathcal{D}(\zeta) |\vec{a}, \vec{Y}\rangle &= \prod_{q=1}^N \Omega(\zeta, a_q, Y_q) |\vec{a}, \vec{Y}\rangle, \\ \Omega(\zeta, a, Y) &\equiv \prod_{\mu \in Y} \frac{(1 + \zeta(a + c(\mu) + 1))(1 + \zeta(a + c(\mu) - \beta))(1 + \zeta(a + c(\mu) - \xi))}{(1 + \zeta(a + c(\mu) - 1))(1 + \zeta(a + c(\mu) + \beta))(1 + \zeta(a + c(\mu) + \xi))}. \end{aligned} \quad (3.18)$$

The factor $\Omega(\zeta, a, Y)$ takes simpler form when a Young diagram Y is divided into rectangles. The building block is the factor for a rectangle $Y = (m)^n$ (a rectangle with m rows and n columns), after some cancellation of factors, we have

$$\Omega(\zeta, a, (m)^n) = \frac{T(\zeta, a + n\beta)T(\zeta, a - m)}{T(\zeta, a)T(\zeta, a + n\beta - m)}. \quad (3.19)$$

Then we may express it as, with the rectangle decomposition of a Young diagram $Y = (r_t, s_t)_{t=1}^f$ ($s_1 > \dots > s_f > s_{f+1} = 0, 0 = r_0 < r_1 < \dots < r_f$),

$$\Omega(\zeta, a, Y) = \frac{1}{T(\zeta, a)} \frac{\prod_{t=1}^{f+1} T(\zeta, a + A_t(Y) + \xi)}{\prod_{t=1}^f T(\zeta, a + B_t(Y))}. \quad (3.20)$$

The eigenvalue for the generating functional $E(\zeta)$ is written in a compact form,

$$\begin{aligned} E(\zeta) |\vec{a}, \vec{Y}\rangle &= \left(\prod_{p=1}^N \Delta(\zeta, a_p, Y_p) \right) |\vec{a}, \vec{Y}\rangle, \\ \Delta(\zeta, a, Y) &= T(\zeta, a) \Omega(\zeta, a, Y) = \frac{\prod_{t=1}^{f+1} T(\zeta, a + A_t(Y) + \xi)}{\prod_{t=1}^f T(\zeta, a + B_t(Y))}. \end{aligned} \quad (3.21)$$

4 Description of minimal models in SH^c

In this section, we first explain the translation rule between the representation of \mathcal{W} -algebra and SH^c . For the minimal model the parameters \vec{a} in SH^c need to take a special value. For such choice of parameters, the Hilbert space of SH^c is supposed to be the same as that of \mathcal{W}_N with an extra $U(1)$ factor where the extra part is realized as the Fock space spanned by one free boson. We show that this is indeed the case by studying the structure of null states in the Hilbert space are identical. We will also come back to the similar setup in section 8 and provide other characterization of \mathcal{W} module, N -Burge condition through SH^c .

4.1 Correspondence between parameters in SH^c and minimal models

The representation of SH^c in (3.3) is parametrized by the central charges, which are written in terms of the vector $\vec{a} \in \mathbb{C}^N$. On the other hand, the minimal model of \mathcal{W}_N -algebra is parametrized by a sequence of integers, which in general labels the momentum of scalar fields in Toda field theory. The vertex operator with a momentum \vec{p} in the conformal Toda theory has the conformal dimension,

$$\Delta(\vec{p}) = \frac{\vec{p} \cdot (\vec{p} - 2Q\vec{\rho})}{2}. \quad (4.1)$$

Comparing with the weight formula (2.2), we may express \vec{p} by (n, n') as follows,

$$\vec{p} = \sum_{i=1}^{N-1} \left(\beta^{-1/2} n_i - \beta^{1/2} n'_i \right) \vec{\omega}_i + Q\vec{\rho} = \sum_{i=1}^{N-1} \left(\beta^{-1/2} \tilde{n}_i - \beta^{1/2} \tilde{n}'_i \right) \vec{\omega}_i, \quad (4.2)$$

with $\beta = q/p$. The parameter \vec{a} in the representation of SH^c is related to \vec{p} as [6]

$$a_i = -\sqrt{\beta} p_i + i\xi. \quad (4.3)$$

We note that \vec{a} and $\vec{a}^\epsilon := (a_{\epsilon(1)}, \dots, a_{\epsilon(N)})$ gives the same representation of SH^c for $\epsilon \in \mathfrak{S}_N$ since the structure constants are symmetric. So one may set \vec{a} by $a_{\epsilon(i)} = -\sqrt{\beta} p_i + i\xi$. Since the inner product (2.4) is symmetric under $i \leftrightarrow N-i$, we have another way to connect the momentum and (n, n') , given by

$$\vec{p}' = \sum_{i=1}^{N-1} \left(\beta^{-1/2} \tilde{n}_{N-i} - \beta^{1/2} \tilde{n}'_{N-i} \right) \vec{\omega}_i. \quad (4.4)$$

In the following the order reversed version will be sometimes useful, which is denoted by

$$a'_{N-i} = -\sqrt{\beta} p'_i + i\xi, \quad (4.5)$$

with $a'_0 = a'_N$. This identification will appear in the next section. Note again that the order of indices in \vec{a}' is not important.

For the particular representations which appear in the level-rank duality, we have $\tilde{n}'_i = 0$ for $1 \leq i \leq N-1$. Also, one should put the parameters as

$$\beta = \frac{N+M}{N}, \quad Q = \frac{M}{\sqrt{(N+M)N}} = \sqrt{\frac{M+N}{N}} - \sqrt{\frac{N}{N+M}}, \quad \xi = -\frac{M}{N}. \quad (4.6)$$

Then some formulae are simplified, and we have

$$\vec{a} = - \sum_{i=1}^{N-1} \tilde{n}_i \vec{\omega}_i + \xi \vec{I}, \quad (4.7)$$

where $\vec{I} = (1, 2, \dots, N)$.

4.2 Singular vectors of minimal models in SH^c

As an application of the correspondence between the parameters in \mathcal{W}_N -algebra and SH^c , let us make a comparison between the structure of null states in the Hilbert space for the general minimal models.

In the case of \mathcal{W}_N -algebra, in the highest weight module with (2.2), there are N null states at level $n_i n'_i$ ($1 \leq i \leq N$). We show that exactly the same null states appear in the SH^c algebra. The null states are characterized by the condition,

$$D_{-1,\ell} |\vec{a}, \vec{Y}\rangle = 0, \quad \ell \geq 0. \quad (4.8)$$

This condition can be satisfied when the factor $\Lambda_p^{(k,-)}(\vec{a}, \vec{Y})$ in (3.17) vanishes for arbitrary p, k . It occurs when at least one numerator factor in (3.17) vanishes for any p, k .

To see null states, first we determine differences among $(a_q)_{q=1}^N$. Let $(\vec{e}_i)_{i=1}^N$ be the standard basis of \mathbb{R}^N . The fundamental weight is determined by the relation $\vec{\omega}_i \cdot \vec{\alpha}_j = \delta_{i,j}$ with $\vec{\alpha}_i = \vec{e}_i - \vec{e}_{i+1}$. We find that

$$\begin{aligned} a_{j+1} - a_j &= -(\vec{\alpha}_j, \vec{a}) = n_j - \beta n'_j, \quad (1 \leq j \leq N-1), \\ a_1 - a_N &= n_N - \beta n'_N, \end{aligned} \quad (4.9)$$

where we used $n_N = N + M - \sum_{i=1}^{N-1} n_i$ and $n'_N \equiv N - \sum_{i=1}^{N-1} n'_i$.

We consider the case when $Y_q = \emptyset$ ($q \neq j+1$) and $Y_{j+1} = (n_j)^{n'_j}$ (a rectangle Young diagram with height n_j and width n'_j). For this case, there is only one box which can be removed with the action of $D_{-1,l}$ and we have $A_1(\emptyset) + \xi = 0$ and $B_1((n_j)^{n'_j}) = \beta n'_j - n_j$. Then a factor $a_{j+1} - a_j + B_1(Y_{j+1}) - A_1(Y_j) - \xi$ in $\Lambda_{j+1}^{(1,-)}$ vanishes. It implies that the condition (4.8) is satisfied for N states specified by,

$$\{\vec{Y}\} = \{(n_N)^{n'_N}, \emptyset, \dots, \emptyset), (\emptyset, (n_1)^{n'_1}, \emptyset, \dots, \emptyset), \dots, (\emptyset, \dots, \emptyset, (n_{N-1})^{n'_{N-1}})\}. \quad (4.10)$$

The levels of these states are $n_i n'_i$ ($1 \leq i \leq N$) which coincide with the null states in \mathcal{W}_N -algebra.⁶ This observation agrees with our expectation that the representation of SH^c with an N -tuple Young diagrams is identical to the representation of \mathcal{W}_N -algebra with the extra $U(1)$ factor which is not relevant in the degeneracy of the Hilbert space.

These null states may be interpreted as the highest weight states with the parameter (\vec{n}, \vec{n}') shifted. Let us focus on the case where $Y_{j+1} = (n_j)^{n'_j}$ and other Y_q 's are \emptyset . This

⁶In [18, 19], the singular vectors in the Virasoro module are identified with the Jack polynomials that correspond to the rectangle Young diagrams. Our statement here is a generalization of such observation since AFLT basis may be regarded as a generalization of Jack polynomials [7].

state has the eigenvalue of $E(\zeta)$ which (3.21):

$$\begin{aligned} & \left(\prod_{q \neq j+1} T(\zeta, a_q) \right) \frac{T(\zeta, a_{j+1} + n'_j \beta) T(\zeta, a_{j+1} - n_j)}{T(\zeta, a_{j+1} + n'_j \beta - n_j)} \\ &= \left(\prod_{q \neq j, j+1} T(\zeta, a_q) \right) T(\zeta, a_{j+1} - n_j) T(\zeta, a_j + n_j) = \mathcal{C}_N(\zeta, \vec{b}). \end{aligned} \quad (4.11)$$

What is interesting here is that the eigenvalue is written as a product of N factors of the form of $\mathcal{C}_N(\zeta, \vec{b})$ with,

$$b_q = \begin{cases} a_{j+1} - n_j & q = j+1 \\ a_j + n_j & q = j \\ a_q & \text{otherwise.} \end{cases} \quad (4.12)$$

So the eigenvalues of the excited state are identical to those of ground state after the parameter shift.

We note that the singular vectors are constructed through the screening operators in the \mathcal{W}_N algebra. In SH^c , on the other hand, they are already built in the coefficients of the representation (3.17).

5 Level-rank duality in SH^c : central charges

We come back to the level-rank duality in the SH^c algebra. The focus of this section is to prove the central charges $\mathcal{C}(\zeta)$ in SH^c are identical for the dual pair. We have seen in (3.6), (3.10) that the other parts of the algebra are identical after suitable rescaling of parameters and generators. On the other hand, $\mathcal{C}(\zeta)$ is kept invariant by the tuning infinite parameters c_n . In the rank N representation, they are replaced by finite parameters a_p and it is not obvious if $\mathcal{C}(\zeta)$ can be kept invariant. If they are identical, it establishes the triality of SH^c for the minimal model CFTs.

The main statement of the section is the following; *with the correspondence between \tilde{n} and \tilde{m} appeared in subsection 2.4, the central charges of SH^c are identical after rescaling (3.6):*

$$\mathcal{C}_N(\zeta, \vec{a}) = \prod_{q=1}^N T(\zeta, a_q) = \prod_{q=1}^M T(\zeta', a'_q) = \mathcal{C}_M(\zeta', \vec{a}'). \quad (5.1)$$

We have to be careful in the definition of a' in the dual pair which will be explained in the proof (5.3).

Proof. First we prepare the explicit form of a_q and a'_q in the main statement (5.1). We recall that, with $\tilde{n}_N = M - \sum_{i=1}^{N-1} \tilde{n}_i$, we rewrite (4.7) as,

$$\begin{aligned} a_i &= \sum_{j < i} \frac{j}{N} \tilde{n}_j - \sum_{j \geq i} \frac{N-j}{N} \tilde{n}_j - \frac{iM}{N} \\ &= \sum_{j < i} \frac{j-i}{N} \tilde{n}_j - \sum_{j \geq i} \frac{N-(j-i)}{N} \tilde{n}_j. \end{aligned} \quad (5.2)$$

Similarly, we rewrite (4.5) and get

$$a'_i = \sum_{j>i}^N \frac{i-j}{N} \tilde{n}_j - \sum_{j\leq i}^N \frac{N-(i-j)}{N} \tilde{n}_j. \quad (5.3)$$

Then we have

$$\begin{cases} a_{j+1} - a_j = \tilde{n}_j + \xi, \\ a_1 - a_N = \tilde{n}_N + \xi, \end{cases} \quad (5.4)$$

$$\begin{cases} a'_j - a'_{j-1} = -\tilde{n}_j - \xi, \\ a'_1 - a'_N = -\tilde{n}_1 - \xi. \end{cases} \quad (5.5)$$

Here we reuse the notation in subsection 2.4. We denote by $(\tilde{n}_i)_{i=1}^L$ the subsequence of all nonvanishing \tilde{n}_i 's as well as by $(\tilde{m}_i)_{i=1}^L$, allocated so that the block with \tilde{m}_i appears just after the one with \tilde{n}_i in the “clock” (see figure 4 in subsection 2.4). We denote by $j(\check{i})$ the labeling satisfying $\tilde{n}_{\check{i}} = \tilde{n}_{j(\check{i})}$ and that the block with $\tilde{n}_{j(\check{i}+1)}$ is the next unoccupied block to the one with $\tilde{n}_{j(\check{i})}$. We also denote by $k(\check{i})$ for $(\tilde{m}_i)_{i=1}^L$, which satisfies $\tilde{m}_{\check{i}} = \tilde{m}_{k(\check{i})}$.

Recall that $a_{j+1} - \xi = a_j$ when $\tilde{n}_j = 0$, and then we have

$$\mathcal{C}_N(\zeta, \vec{a}) = \prod_{\check{i}=1}^L \frac{1 + \zeta a_{j(\check{i})}}{1 + \zeta(a_{j(\check{i}+1)} - \xi)} = \prod_{\check{i}=1}^L \frac{1 + \zeta a_{j(\check{i})}}{1 + \zeta(a_{j(\check{i})} + \tilde{n}_{\check{i}})}. \quad (5.6)$$

Also we have

$$\mathcal{C}_M(\zeta', \vec{a}') = \prod_{\check{i}=1}^L \frac{1 + \zeta' a'_{k(\check{i})}}{1 + \zeta'(a'_{k(\check{i})-1} - \xi)} = \prod_{\check{i}=1}^L \frac{1 + \zeta' a'_{k(\check{i})}}{1 + \zeta'(a'_{k(\check{i})} + \tilde{m}_{\check{i}})}. \quad (5.7)$$

A similar discussion in subsection 2.4 leads to

$$\begin{aligned} Na_{j(\check{i})} &= - \sum_{\check{j}<\check{i}} \sum_{\check{j}\leq\check{k}<\check{i}} \tilde{m}_{\check{k}} \tilde{n}_{\check{j}} - \sum_{\check{j}\geq\check{i}} \sum_{\check{j}\leq\check{k}<\check{i}+L} \tilde{m}_{\check{k}} \tilde{n}_{\check{j}} \\ &= - \sum_{\check{k}<\check{i}} \sum_{\check{i}\leq\check{j}\leq\check{k}+L} \tilde{m}_{\check{k}} \tilde{n}_{\check{j}} - \sum_{\check{k}\geq\check{i}} \sum_{\check{i}\leq\check{j}\leq\check{k}} \tilde{m}_{\check{k}} \tilde{n}_{\check{j}}, \\ Ma'_{k(\check{i})} &= - \sum_{\check{k}>\check{i}} \sum_{\check{i}<\check{j}\leq\check{k}} \tilde{m}_{\check{k}} \tilde{n}_{\check{j}} - \sum_{\check{k}\leq\check{i}} \sum_{\check{i}<\check{j}\leq\check{k}+L} \tilde{m}_{\check{k}} \tilde{n}_{\check{j}}. \end{aligned} \quad (5.8)$$

Therefore

$$Na_{j(\check{i}+1)} = Ma'_{k(\check{i})}. \quad (5.9)$$

Note that

$$Na_{j(\check{i}+1)} - Na_{j(\check{i})} = N\tilde{n}_{\check{i}} - M\tilde{m}_{\check{i}}, \quad (5.10)$$

then

$$M(a'_{k(\check{i})} + \tilde{m}_{\check{i}}) = N(a_{j(\check{i})} + \tilde{n}_{\check{i}}). \quad (5.11)$$

As a result, we have $\mathcal{C}_N(\zeta, \vec{a}) = \mathcal{C}_M(\zeta', \vec{a}')$. \square

Example. To illustrate the statement of the section, we demonstrate the explicit form of $\mathcal{C}_N(\zeta)$ for $(N, M) = (2, 3)$.

For the $\Delta = 0$ case, on the $N = 2$ side, we take $\tilde{n} = (0, 3)$ and then we have

$$\{2\vec{a}_i\} = \{-3, -6\}, \quad \{2(a_i - \xi)\} = \{0, -3\}. \quad (5.12)$$

Therefore, after the cancellation between the numerator and denominator, we have

$$\mathcal{C}_2(\zeta, \vec{a}) = 1 - 6 \cdot \frac{\zeta}{2}. \quad (5.13)$$

On the $M = 3$ side, the same state is described by $\tilde{m} = (0, 0, 2)$. We obtain,⁷

$$\{3a'_i\} = \{-2, -4, -6\}, \quad \{3(a'_i - \xi)\} = \{0, -2, -4\}. \quad (5.14)$$

There is again a similar cancellation and we obtain

$$\mathcal{C}_3(\zeta', \vec{a}') = 1 - 6 \cdot \frac{\zeta'}{3}. \quad (5.15)$$

After the identification $\zeta/2 = \zeta'/3$, we obtain (5.1).

For the $\Delta = -1/5$ case, on the $N = 2$ side, we need to take $\tilde{n} = (1, 2)$, which implies $\{2a_i\} = \{-4, -5\}$, $\{2(a_i - \xi)\} = \{-1, -2\}$ and

$$\mathcal{C}_2(\zeta, \vec{a}) = \frac{(1 - 4\zeta/2)(1 - 5\zeta/2)}{(1 - \zeta/2)(1 - 2\zeta/2)}. \quad (5.16)$$

In the dual picture, $\tilde{m} = (0, 1, 1)$ and $\{3a'_i\} = \{-3, -4, -5\}$ and $\{3(a'_i - \xi)\} = \{-1, -2, -3\}$. Again after the cancellation we obtain

$$\mathcal{C}_3(\zeta', \vec{a}') = \frac{(1 - 4\zeta'/3)(1 - 5\zeta'/3)}{(1 - \zeta'/3)(1 - 2\zeta'/3)}. \quad (5.17)$$

In either case some cancellations between the numerator and denominator occur and the numbers of factors match in the end while the dimension of \vec{a} is different.

6 Level-rank duality in SH^c : state-to-state correspondence

We have proved that the central charges of SH^c are identical for the pair of the minimal models with the level-rank duality. If the two algebras are identical, so should be the representations. It remains to be curious, however, how the states labeled by different sets of Young diagrams can be identified. Of course, it is possible only when some of the basis become null.

We note that in order to identify two states labeled by different sets of Young diagrams, the eigenvalues of $D_{0,l}$ should match. Eq. (3.16) implies that the eigenvalue is the power sum of the numbers $a_q + c(\mu)$ assigned to each box in \vec{Y} . The transformation of $D_{0,l}$ under

⁷Here we write elements in the decreasing order. Recall that there is an arbitrariness of arrangement of indices in a_i, a'_j .

σ_2 (3.6) becomes, $D'_{0,l+1} = (N/M)^l D_{0,l+1}$. By combining them, we need to require the rescaled numbers assigned to each boxes in \vec{Y} and \vec{Y}' ,

$$\sqcup_{q=1}^N \{N(a_q + c(\mu))\}_{\mu \in Y_q} = \sqcup_{q=1}^M \{M(a'_q + c(\mu'))\}_{\mu' \in Y'_q} \quad (6.1)$$

to be equal as a set in order to identify $|\vec{a}, \vec{Y}\rangle$ and $|\vec{a}', \vec{Y}'\rangle$. We refer to $N(a_q + c(\mu))$ as the *characterizing number* of the box. With the help of such observations, the level-rank duality is realized as an intertwining map which shuffles rows of Young diagrams appropriately. In order to have such consistent identification of boxes in \vec{Y} and \vec{Y}' , we take the following steps.

Step 1: labels of the rows. We define the set of the characterizing numbers of the leftmost boxes on each Young diagrams as X ,

$$\begin{aligned} X &= \sqcup_{q=1}^N \sqcup_{j=1}^\infty \{N(a_q + c(1, j))\} = \sqcup_{q=1}^N \sqcup_{j=1}^\infty \{Na_q + M - N(j-1)\} \\ X' &= \sqcup_{q=1}^M \sqcup_{j=1}^\infty \{M(a'_q + c(1, j))\} = \sqcup_{q=1}^M \sqcup_{j=1}^\infty \{Ma'_q + N - M(j-1)\}. \end{aligned} \quad (6.2)$$

We claim that they have the following properties:

(1-1) In either set X or X' , there is no overlap of elements.

(1-2) If $x \in X$ (resp. X') then $x - N, x - M \in X$ (resp. X').

(1-3) Two sets are identical: $X = X'$.

These statements are nontrivial and will be proved in subsection 6.2. They imply that there is a one-to-one correspondence between the leftmost boxes in \vec{Y} and \vec{Y}' . We use the number $x \in X$ to label the rows of Young diagrams.

Step 2: correspondence between states. Once we have the identification of leftmost boxes, one may assign the correspondence between the boxes in the same rows at the same time. Suppose $(1, j) \in Y_q$ and $(1, j') \in Y'_{q'}$ to have the same label x . Then both boxes, $(i, j) \in Y_q$ and $(i, j') \in Y'_{q'}$ have the same characterizing number $x + (N + M)(i - 1)$. We establish the correspondence between the rows with the same label since they do not violate (6.1).

It implies that the two states $|\vec{a}, \vec{Y}\rangle$ and $|\vec{a}', \vec{Y}'\rangle$ are identified if the length of rows with the same label are identical (see figure 6). In general, two rows which belong to the same Y_q may not correspond to two rows in the same $Y'_{q'}$ in \vec{Y}' . We call such restructuring of Young diagrams as the *shuffling of rows*.

For the consistency of such identification, the shapes of Young diagrams in \vec{Y} or \vec{Y}' are constrained. In order that each Y_q is the Young diagram, we need to have an inequality $\lambda_{\vec{Y}}(x - N) \leq \lambda_{\vec{Y}}(x)$ for each $x \in X$ where $\lambda_{\vec{Y}}(x)$ denotes the number of boxes in the row characterized by the integer $x \in X$ with respect to \vec{Y} . The mapped shapes form a vector of Young diagrams on the dual side if and only if $\lambda_{\vec{Y}}(x - M) \leq \lambda_{\vec{Y}}(x)$ is satisfied for each $x \in X$.

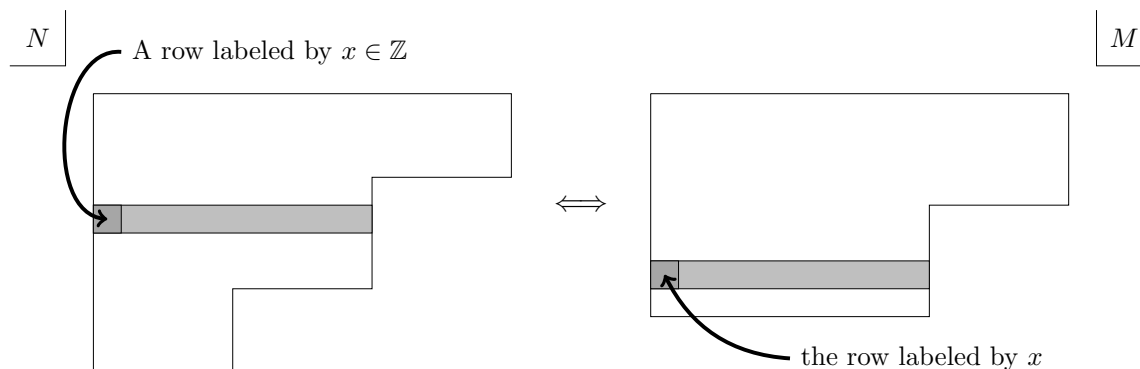


Figure 6. Rearrangement of Young diagrams with the level-rank duality. Two gray rows have the same number of boxes.

Finally we arrive at the main statement of the section:

- (2) For the dual pair of the minimal models, the excited states labeled by either \vec{Y} or \vec{Y}' are obtained from a single set of the integer partitions $\lambda : X \rightarrow \mathbb{N}$, $|\lambda| \equiv \sum_{x \in X} \lambda(x) < \infty$, satisfying the condition $\lambda(x - N), \lambda(x - M) \leq \lambda(x)$ for each $x \in X$. Then the shuffling of rows gives an automorphism between the two representations.

Drawing an analogy with Young diagrams, we introduce a partial order $<_X$ to the set X by

$$x \geq_X y \Leftrightarrow \exists i_N, i_M \in \mathbb{N}, x = y - Ni_N - Mi_M. \quad (6.3)$$

Then we rewrite the condition for the partition for the partially ordered set (poset) by

$$x \geq_X y \Rightarrow \lambda_{\vec{Y}}(x) \leq \lambda_{\vec{Y}}(y). \quad (6.4)$$

In subsection 6.3, we will show that the module obtained by applying $D_{1,l}$ to the vacuum state $|\emptyset, \dots, \emptyset\rangle$ are spanned by all such restricted states. However we should recall that we have taken a_q 's to specific values for the minimal models. It may cause divergence of Λ^\pm in (3.17). It triggers us to check that the highest weight representation is well-defined for the case of the level-rank duality. We will check it by deriving the condition when the factors in the product in (3.17) vanish. This will in turn lead to the following statement:

- (3) The highest weight representation is not only well-defined but also is spanned by *all* the integer partition over X satisfying (6.4).

Then the claim (2) will follow as a corollary of the statement (3).

6.1 Example

Since the main statements of this section may be abstract, it is illuminating to describe them in concrete examples. The case $(N, M) = (2, 3)$ is the easiest nontrivial example. We have two primary fields with conformal weights $\Delta = 0$ and $\Delta = -1/5$.

For $\Delta = 0$ case, we have $\vec{a} = \{-3/2, -3\}$ and $\vec{a}' = \{-2/3, -4/3, -2\}$. The set X, X' of labels are computed as,

$$X = X' = \{0, -2, -3, -4, -5, -6, \dots\}. \quad (6.5)$$

For $\Delta = 1/5$, similar computation gives,

$$X = X' = \{-1, -2, -3, -4, -5, -6, \dots\}. \quad (6.6)$$

In either case, X, X' satisfy the properties claimed in step 1.

Let us examine what states are generated by applying $D_{1,l}$ to the vacuum. We focus on the one with $\Delta = -1/5$ here. We use the parameter $\vec{a} = (-2, -5/2) \leftrightarrow \vec{a}' = (-4/3, -1, -5/3)$ but omit them in the label of excited states. The labels for the Young diagrams (recall figure 1) are

$$N=2 \quad Y_1 \leftrightarrow \{-1, -3, -5, \dots\}, \quad Y_2 \leftrightarrow \{-2, -4, -6, \dots\} \quad (6.7)$$

$$M=3 \quad Y'_1 \leftrightarrow \{-2, -5, -8, \dots\}, \quad Y'_2 \leftrightarrow \{-1, -4, -7, \dots\}, \quad Y'_3 \leftrightarrow \{-3, -6, -9, \dots\}. \quad (6.8)$$

At level one (one Young box),

$$\begin{aligned} D_{1,l} |\emptyset, \emptyset\rangle &= (-1)^l \left(\left(-\frac{1}{2}\right)^l \times 2 |\square, \emptyset\rangle + (-1)^l \times i\sqrt{2} |\emptyset, \square\rangle \right), \\ D_{1,l} |\emptyset, \emptyset, \emptyset\rangle &= (-1)^l \left(\left(-\frac{2}{3}\right)^l \times i\sqrt{3} |\square, \emptyset, \emptyset\rangle + \left(-\frac{1}{3}\right)^l \times \sqrt{6} |\emptyset, \square, \emptyset\rangle \right). \end{aligned} \quad (6.9)$$

After required rescaling $D_{1,l} \mapsto \left(\frac{2}{3}\right)^{l-1/2} D_{1,l}$, we can identify,

$$|\square, \emptyset\rangle \leftrightarrow |\emptyset, \square, \emptyset\rangle \quad |\emptyset, \square\rangle \leftrightarrow |\square, \emptyset, \emptyset\rangle. \quad (6.10)$$

The two states correspond to the partition $\lambda(-1) = 1, \lambda(-2) = \lambda(-3) = \dots = 0$ and $\lambda(-2) = 1, \lambda(-1) = \lambda(-3) = \lambda(-4) = \dots = 0$. We note that a state $|\emptyset, \emptyset, \square\rangle$ is not created. It corresponds to a partition $\lambda(-3) = 1, \lambda(-1) = \lambda(-2) = \lambda(-4) = \dots = 0$ but it breaks the rule (6.4) since $\lambda(-1) < \lambda(-3)$.

We can perform similar such computation to higher levels in the same way. The correspondence between non-null states in the Lee-Yang case will be explicitly given up to level 5 in appendix A. It is easy to see the shuffling rule of rows and the condition (6.4) are satisfied in all of them.

6.2 Proof of Step 1

Here we give a proof of the statements in Step 1.

We first confirm that there is no overlap of elements in the set,

$$X = \{N(a_q + c(1, s)) | q = 1, \dots, N, s \geq 1\} \quad (6.11)$$

namely the statement (1-1). Actually this is obvious since we have $N(a_q + c(1, s)) = Na_q + M - N(s-1)$ and we see from (5.4) that no two integers Na_q 's modulo N are the same.

Next we prove the statement (1-2),

$$x \in X \Rightarrow x - N, x - M \in X. \quad (6.12)$$

It is sufficient to show that for $x \in \{Na_q + M\}_{q=1}^N$, $x + M \in X$. Now we rewrite the subset as

$$\begin{aligned} \{Na_q + M\}_{q=1}^N &= \sqcup_{i=1}^L \{N(a_{j(\check{i})} + \check{n}_{\check{i}}) - M(l-1) \mid 1 \leq l \leq \check{m}_{\check{i}}\}, \\ \{Ma'_q + N\}_{q=1}^M &= \sqcup_{i=1}^L \{M(a'_{k(\check{i})} + \check{m}_{\check{i}}) - N(l-1) \mid 1 \leq l \leq \check{n}_{\check{i}}\} \end{aligned} \quad (6.13)$$

by use of the notations appeared in the previous section. Then the following equations shows the property;

$$\begin{aligned} (N(a_{j(\check{i})} + \check{n}_{\check{i}}) - M(\check{m}_{\check{i}} - 1)) - M &= Na_{j(\check{i}+1)} = N(a_{j(\check{i}+1)} + \check{n}_{\check{i}+1}) - N\check{n}_{\check{i}+1} \in X, \\ (M(a'_{k(\check{i})} + \check{m}_{\check{i}}) - N(\check{n}_{\check{i}} - 1)) - N &= Ma'_{k(\check{i}-1)} = M(a'_{k(\check{i}-1)} + \check{m}_{\check{i}-1}) - M\check{m}_{\check{i}-1} \in X'. \end{aligned} \quad (6.14)$$

Finally, the equation (5.11) means that the local maxima of X are identical to those of X' , and then the property (1-2) shows that $X = X'$, the statement (1-3).

6.3 Proof of Step 2

Difference of the characterizing numbers. Under the condition (6.4), we consider the difference of characterizing numbers $N(a_q + c(1, s))$ of two boxes both located on the rightmost edges of Young diagrams. The number for either box has of the form

$$x + (\lambda_{\vec{Y}}(x) - 1)(N + M), \quad (6.15)$$

and their difference becomes

$$(x - y) + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y))(N + M). \quad (6.16)$$

It does not vanish if $x <_X y$ or $y <_X x$ by the definition of $<_X$. Suppose that the factor (6.16) vanishes, it implies

$$y = x + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y))N + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y))M, \quad (6.17)$$

which leads to either $x \leq_X y$ or $x \geq_X y$. We can see the only consistent solution is $x = y$. This result implies the factors $a_p - a_q + A_k(Y_p) - A_\ell(Y_q)$ and $a_p - a_q + B_k(Y_p) - B_\ell(Y_q)$ in the denominator of Λ^\pm , (3.17), vanish if and only if for $(p, k) = (q, l)$.

Similarly, if the factor

$$(x - y) + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y))(N + M) + (N + M) = (x - y) + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y) + 1)(N + M) \quad (6.18)$$

vanishes, we can only have $x <_X y$ or $x >_X y$, and then the condition (6.4) gives $x >_X y$ and $\lambda_{\vec{Y}}(x) = \lambda_{\vec{Y}}(y)$. As a result, if (6.18) vanishes, we have $y = x + N + M$ and $\lambda_{\vec{Y}}(x) = \lambda_{\vec{Y}}(y)$. In particular, we have $\lambda_{\vec{Y}}(x) = \lambda_{\vec{Y}}(x + N)$. This, in turn, implies $a_p - a_q + A_k(Y_p) - B_\ell(Y_q)$ in the product (3.17) does not vanish since the factor $a_p + A_k(Y_p)$ comes from a row x with $\lambda_{\vec{Y}}(x) < \lambda_{\vec{Y}}(x + N)$.

On the other hand, the factor $a_p - a_q + A_k(Y_p) - B_\ell(Y_q) + \xi$ in (3.17) corresponds to

$$(x - y) + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y))(N + M) + N. \quad (6.19)$$

Through the same discussion, the only possibility for this factor to vanish reduces to $y = x + N$ and $\lambda_{\vec{Y}}(x) = \lambda_{\vec{Y}}(x + N)$. Therefore $a_p - a_q + A_k(Y_p) - B_\ell(Y_q) + \xi$ never vanishes.

As a result, the Λ^+ (resp. Λ^-) in (3.17) vanishes if and only if an upper-right factor $a_p - a_q + A_k(Y_p) - A_\ell(Y_q) + \xi$ (resp. $a_p - a_q + B_k(Y_p) - B_\ell(Y_q) + \xi$) vanishes. Note that these factors correspond to

$$(x - y) + (\lambda_{\vec{Y}}(x) - \lambda_{\vec{Y}}(y))(N + M) + M, \quad (6.20)$$

which vanishes if $y = x + M$ and $\lambda_{\vec{Y}}(x) = \lambda_{\vec{Y}}(x + M)$, which can be shown by the same discussion.

Proof of the statement (3). We show that no vector of Young diagrams which breaks the condition (6.4) appears in the highest weight representation. This means that, for the finiteness of Λ^\pm , the highest weight representation is well-defined. Also, it shows that the module is spanned by *some of* the integer partitions satisfying (6.4).

Let \vec{Y} be an N -tuple Young diagrams whose $\lambda_{\vec{Y}}$ satisfies (6.4). Then let us denote \vec{Y}_+ be another N -tuple obtained by adding one box to \vec{Y} at the position $a_p + A_k(Y_p)$. Assume that $\lambda_{\vec{Y}_+}$ does not satisfy the condition (6.4). Then we should have $\lambda_{\vec{Y}}(x + M) = \lambda_{\vec{Y}}(x)$ for $x = N(a_p + c(1, s_k^{(p)} + 1))$ such that $x + M \in X$, where $Y_p = (r_t^{(p)}, s_t^{(p)})_{t=1}^{f_p}$ is the rectangle decomposition of Y_p . Since the condition (6.4) holds for the other rows, we see that in the row with $x + M$ we can add a box to obtain another N -tuple Young diagrams. In other words, it is a hollow place of the Young diagram belonging to the row with $x + M$. This means that there exist $q \neq p$ and l such that $N(a_q + A_l(Y_q)) = N(a_p + A_k(Y_p)) + M$. This equation forces a factor in the numerator of $\Lambda_p^{(k,+)}(\vec{a}, \vec{Y})$ to vanish, which prevents $|\vec{a}, \vec{Y}_+\rangle$ to appear in the representation. Therefore, the action of $D_{1,l}$ closes in the subspace spanned by such desired vectors. So is $D_{-1,l}$, shown by a similar discussion.

We complete the proof of the statement (3). What remains is that *all* the states labeled by integer partitions satisfying (6.4) appear in the highest weight representation. However, it is immediate as follows. With the notations above, assume that $\lambda_{\vec{Y}_+}$ also satisfies (6.4), and then $\Lambda_q^{(k,\pm)}(\vec{a}, \vec{Y})$ does not vanish for $\lambda_{\vec{Y}}(x + M) > \lambda_{\vec{Y}}(x)$ and a discussion around (6.20). Since no two arbitrary $a_q + A_l(Y_q)$ are identical, then an appropriate linear combination of $D_{1,l}$ maps $|\vec{a}, \vec{Y}\rangle$ to $|\vec{a}, \vec{Y}_+\rangle$.

Proof of the statement (2). Finally we would like to establish that the action of $D_{\pm 1,l}$ is consistent with the shuffling of rows. We note that we can rewrite the products appeared in the definition (3.17) of Λ^\pm to some products over all the rows but one^{8,9} (see figure 7). This implies that $D_{\pm 1,l}$ and the shuffling commute up to an overall rescaling (3.10) of $D_{\pm 1,l}$ with $\gamma = 1$ under the level-rank duality.

⁸The missing one comes from the upper right numerator factor in Λ^\pm , which corresponds to $Y_p = Y_q$ and $k = \ell$. This missing factor is merely a constant $((\pm)M)^{-1/2}$, which reduces to the overall rescaling factor $(M/N)^{-1/2}$.

⁹To avoid discussions about zeroth or poles, one may shift all the factors in the tetrimino by a small parameter h . After the product is done, we take a limit $h \rightarrow 0$ and obtain the original Λ^\pm .

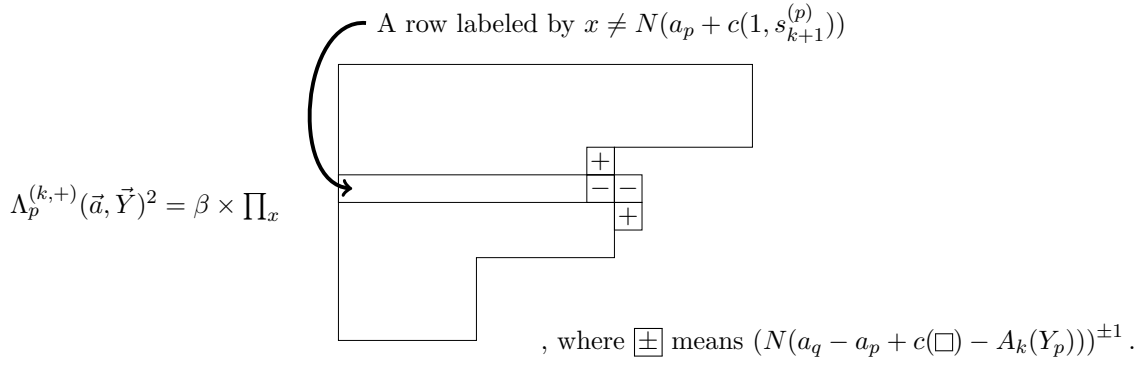


Figure 7. Diagrammatic expression of $\Lambda_p^{(k,+)}(\vec{a}, \vec{Y})$.

In summary, it has been shown that one can obtain the irreducible representation from the level-0 state just by setting parameters of the SH^c representation to fit minimal models with the level-rank duality. Then the level-rank duality is realized as an intertwining map which shuffles rows of Young diagrams, following the characterizing set X .

7 Poset and partition function

For the minimal models with the level-rank duality, we have understood that the excited states in the SH^c module are described by the partitions $\lambda : X \rightarrow \mathbb{N}$. Recall we have introduced a partial order $<_X$ to the set X by

$$x \geq_X y \Leftrightarrow \exists i_N, i_M \in \mathbb{N}, x = y - Ni_N - Mi_M, \quad (7.1)$$

and the partition λ is defined by

$$x \geq_X y \Rightarrow \lambda(x) \leq \lambda(y). \quad (7.2)$$

Denoting the set of all such partitions by $\mathcal{A}(X)$, now we consider the Hilbert series $Z_X(q)$ of the representation graded by its level, which is defined by

$$Z_X(q) = \sum_{\lambda \in \mathcal{A}(X)} q^{|\lambda|}. \quad (7.3)$$

We note that it defines a new way to count the number of states for (the special cases of) the minimal models.

The character of the general minimal model was derived through the coset construction [20] and N-Burge condition [21]. For our special cases, it is given in the infinite product form [11]. While it is well-known, the new counting method may provide a new way to write the character which produces nontrivial identities.

For the set $X = \mathbb{N}$ with the canonical order, the computation of the partition function is elementary:

$$Z_{\mathbb{N}}(q) = \prod_{i=1}^{\infty} \sum_{n_i=0}^{\infty} q^{in_i} = \frac{1}{(q; q)_{\infty}}, \quad (7.4)$$

where $(a; q)_j = \prod_{k=0}^{j-1} (1 - aq^k)$ is the q -Pochhammer symbol.

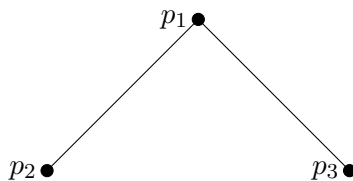


Figure 8. The Hasse diagram of $P = \{p_1, p_2, p_3\}$.

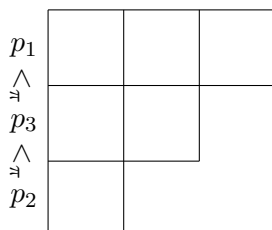


Figure 9. The P-partition $(3, 1, 2)$ expressed by a Young diagram with a total ordering $<_\pi$.

How do we calculate partition function for $Z_X(q)$ for the partially ordered set? In mathematics, the integer partitions on the partially ordered set are well studied and called *P-partitions*. Before we introduce some results in the book [14], we explain how to organize $\mathcal{A}(X)$ to calculate the Hilbert series by dealing with a toy example.

7.1 A toy example

Let us consider a partially ordered set $P = \{p_1, p_2, p_3\}$ of three points as an example, whose ordering is given by $p_1 <_P p_2$ and $p_1 <_P p_3$. Its Hasse diagram¹⁰ is represented as figure 8. We denote by $\mathcal{A}(P)$ the set of partitions $\lambda : P \rightarrow \mathbb{N}$ satisfying $|\lambda| < \infty$ and (7.2) for $X = P$, and call its element a P-partition. For example, $(p_1, p_2, p_3) \mapsto (2, 1, 0), (2, 0, 1)$ are P-partitions.

Note that, for each P-partition λ , there is a total ordering $<_\pi$ of P satisfying

$$x <_P y \Rightarrow x <_\pi y \quad (7.5)$$

and

$$x <_\pi y \Rightarrow \lambda(x) \geq \lambda(y). \quad (7.6)$$

In other words, for given a P-partition, we can arrange all the elements of P into a line compatible with the ordering $<_P$ and then the partition is expressed by a Young diagram whose rows represent the linearized P and $\lambda(x)$ is equal to the number of boxes in the row with x for each $x \in P$. For example, for the P-partition $(p_1, p_2, p_3) \mapsto (3, 1, 2)$, we linearize P by a total order $p_1 <_\pi p_3 <_\pi p_2$ and then the corresponding diagram is expressed by a Young diagram appearing in figure 9.

¹⁰Given a partially ordered set P , its Hasse diagram is defined by assigning a point for each $x \in P$ and drawing a line between two points representing $x, y \in P$ if $x <_P y$ and there is no element $z \in P$ such that $x <_P z <_P y$. Clearly, if $x <_P y$, x and y lies in the same connected component of the graph.

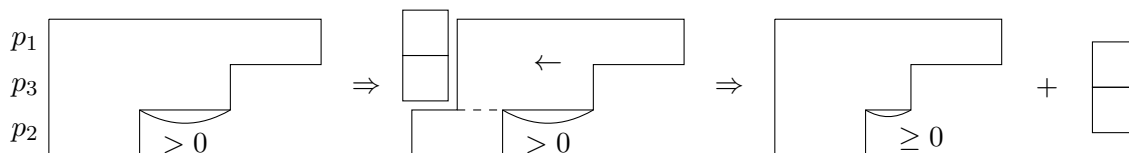


Figure 10. Modification of the condition $\lambda(p_3) > \lambda(p_2)$ in S_2 .

Then, in order to compute the Hilbert series of P , we want to correspond a P-partition to a pair of a total ordering compatible to $<_P$ and a Young diagram. We have to be careful to avoid overcountings. For example, for a P-partition $(p_1, p_2, p_3) \mapsto (2, 1, 1)$, there are two total ordering $p_1 < p_2 < p_3$ and $p_1 < p_3 < p_2$ both of which give the same Young diagram $\begin{smallmatrix} \square & \square \\ \square & \\ \square & \end{smallmatrix}$. To count without overlaps, we fix one total ordering, say $p_1 < p_2 < p_3$, as the canonical one, and correspond that partition to the pair of this canonical ordering and the Young diagram $\begin{smallmatrix} \square & \square \\ \square & \\ \square & \end{smallmatrix}$. As a result, we separate $\mathcal{A}(P)$ into the two set S_1, S_2 :

$$\begin{aligned} \mathcal{A}(P) &= S_1 \sqcup S_2, \\ S_1 &\equiv \{\lambda \in \mathcal{A}(P) \mid \lambda(p_1) \geq \lambda(p_2) \geq \lambda(p_3)\}, \\ S_2 &\equiv \{\lambda \in \mathcal{A}(P) \mid \lambda(p_1) \geq \lambda(p_3) > \lambda(p_2)\}. \end{aligned} \quad (7.7)$$

Now we can compute the Hilbert series through a similar way we performed in (7.4) for the case with $X = \mathbb{N}$. On the S_1 side, we have

$$\sum_{\lambda \in S_1} q^{|\lambda|} = \frac{1}{(q; q)_3}. \quad (7.8)$$

On the other side, through a diagrammatic consideration in figure 10 for the condition $\lambda(p_3) > \lambda(p_2)$ in the definition of S_2 , we have

$$\sum_{\lambda \in S_2} q^{|\lambda|} = q^2 \sum_{\lambda \in S_1} q^{|\lambda|} = \frac{q^2}{(q; q)_3}. \quad (7.9)$$

Then we have $Z_P(q) = (1 + q^2)/(q; q)_3$. Note that the q^2 appeared in the numerator of (7.9) comes from the fact that the p_3 , the 2nd lowest element in P with respect to the ordering $p_1 < p_3 < p_2$, is larger than the next element p_2 with respect to the fixed canonical ordering $p_1 < p_2 < p_3$. In other words, this factor represents where a total ordering of P differs from the canonical total ordering.

7.2 General cases

This example gives us a lesson on how to compute the Hilbert series for a general partially ordered set which is bounded below with finite local minima and whose Hasse diagram is connected:¹¹

¹¹If the Hasse diagram is not connected, the Hilbert series is given by the product of the series for each connected component of the partially ordered set.

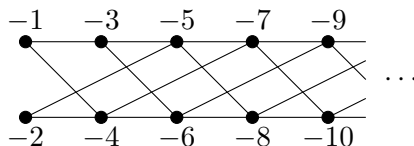


Figure 11. The Hasse diagram of X for $\Delta = -\frac{1}{5}$.

1. Define $\mathcal{L}(X)$ as the set of all the total orderings compatible to the ordering $<_X$, known for mathematicians as the *Jordan-Hölder* set of X .
2. Fix an element of $c \in \mathcal{L}(X)$ as the canonical ordering, and then for each $\pi \in \mathcal{L}(X)$ define

$$S_\pi = \left\{ \lambda \in \mathcal{A}(X) \left| \begin{array}{l} x <_\pi y \Rightarrow \lambda(x) \geq \lambda(y) \\ x >_c x+1 \Rightarrow \lambda(x) > \lambda(x+1) \end{array} \right. \right\}, \quad (7.10)$$

where $x+1$ is the next element to x with respect to the ordering π .

3. Denoting the set of all the positions where two ordering $\pi \in \mathcal{L}(X)$ and c differ by

$$D_\pi = \{ j \in \mathbb{N}^+ \mid x_j >_c x_{j+1} \}, \quad (7.11)$$

where x_j represents the j -th lowest element of X with respect to π , we have¹² [14]

$$\mathcal{A}(X) = \bigsqcup_{\pi \in \mathcal{L}(X)} S_\pi, \quad Z_X(q) = \sum_{\lambda \in \mathcal{A}(X)} q^{|\lambda|} = \frac{\sum_{\pi \in \mathcal{L}(X)} \prod_{j \in D_\pi} q^j}{(q; q)_{|X|}}, \quad (7.12)$$

where $|X|$ is the cardinality of X .

Here we rewrite D_π as follows. For a given $\pi \in \mathcal{L}(X)$, we denote by $\pi(i) = j$ that the i -th lowest element with respect to π is the j -th lowest one with respect to the canonical ordering. Then we have

$$D_\pi = \{ i \mid \pi(i) > \pi(i+1) \}. \quad (7.13)$$

7.3 Application to minimal model: (2, 3) case

We have seen that the computation of the partition function for the poset is reduced to finding the Jordan-Hölder set of X . It is, however, difficult to find it for the poset X for minimal models at this moment. Fortunately, for the simplest example $(N, M) = (2, 3)$, we can explicitly find them and compute $Z_X(q)$ explicitly.

For the $\Delta = -1/5$ case ($\tilde{n} = (1, 2)$), we have $\{Na_q + M\} = \{-1, -2\}$ on the $N = 2$ side. Then we obtain figure 11 as the Hasse diagram of the corresponding ordered set X . Now we fix as the canonical ordering a total ordering where the i -th lowest element is $-i \in X$. Then since π is an element of $\mathcal{L}(X)$, we have

$$\pi(1) = 1, \quad \text{or} \quad (\pi(1), \pi(2)) = (2, 1). \quad (7.14)$$

¹²The corresponding statement in the book [14] is for partially ordered sets with *finite* cardinality. However, it can be enlarged for our infinite cases since boxes are piled up from the bottom.

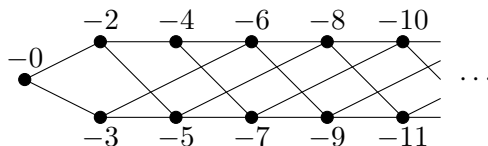


Figure 12. The Hasse diagram for $\Delta = 0$.

In either case, the lowest element or pair with respect to π is selected as the lowest one with respect to the canonical ordering. Note that the subset obtained by the subtraction of a set $\{x \in X | x <_c -j\}$ from X for a given integer j gives the same Hasse diagram¹³ as the one with the original X . Therefore a similar analysis goes well for more higher elements of X , and then we can identify $\mathcal{L}(X)$ with a set of integers

$$\{1 \leq t_1 < t_2 < \dots < t_k \mid k \in \mathbb{N}, t_i - t_{i-1} \geq 2\}, \quad (7.15)$$

where the isomorphism is given by $\pi(t_i) > \pi(t_i + 1)$. Surprisingly, the above set is none other than the one of integer partitions with the Rogers-Ramanujan identities (see [22, 23])! As a result, we have

$$\begin{aligned} Z_X^{(\Delta=-\frac{1}{5})}(q) &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} \sum_{\substack{1 \leq t_1 < \dots < t_k \\ t_i - t_{i-1} \geq 2}} q^{\sum_{j=1}^k t_j} \\ &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} \sum_{0 \leq t_1 \leq \dots \leq t_k} q^{k + \sum_{j=0}^{k-1} 2j + \sum_{j=1}^k t_j} \\ &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} q^{k^2} \sum_{0 \leq t_1 \leq \dots \leq t_k} q^{\sum_{j=1}^k t_j} \\ &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k}. \end{aligned} \quad (7.16)$$

The partition function for this case is known:

$$Z_X^{(\Delta=-\frac{1}{5})}(q) = \frac{1}{(q; q)_\infty (q; q^5)_\infty (q^4; q^5)_\infty} = \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty (q^5; q^5)_\infty}{(q; q)_\infty^2}. \quad (7.17)$$

The partition function obtained by two methods gives rather different form. It is, however, the same due to the first Rogers-Ramanujan identity [24]. In a sense, the computation of partition through the poset suggests Rogers-Ramanujan identity.

We study the other case $\Delta = 0$ ($\tilde{n} = (0, 3)$, $Na_q + M = 0, -3$). The Hasse diagram of the corresponding partially ordered set is given in figure 12. We take the total ordering $0 <_c -2 <_c -3 <_c \dots$ as the canonical one and then we find $\pi(1) = 1$ for any $\pi \in \mathcal{L}(X)$. Note that we see from figure 12 that the Hasse diagram of the subset $X \setminus \{0\}$ is identical to the figure 11. Therefore we can apply the previous discussions to higher elements of X .

¹³This is why we can determine $\mathcal{L}(X)$ for the Lee-Yang edge singularity.

As a result, we identify $\mathcal{L}(X)$ with

$$\{1 < t_1 < t_2 < \cdots < t_k \mid k \in \mathbb{N}, t_i - t_{i-1} \geq 2\}. \quad (7.18)$$

Therefore, we have

$$\begin{aligned} Z_X^{(\Delta=0)}(q) &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} \sum_{\substack{1 < t_1 < \cdots < t_k \\ t_i - t_{i-1} \geq 2}} q^{\sum_{j=1}^k t_j} \\ &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} \sum_{0 \leq t_1 \leq \cdots \leq t_k} q^{2k + \sum_{j=0}^{k-1} 2j + \sum_{j=1}^k t_j} \\ &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_k} \\ &= \frac{1}{(q; q)_\infty (q^2; q^5)_\infty (q^3; q^5)_\infty} = \frac{(q; q^5)_\infty (q^4; q^5)_\infty (q^5; q^5)_\infty}{(q; q)_\infty^2}. \end{aligned} \quad (7.19)$$

We have met the second Rogers-Ramanujan identity [24] when we go to the last line.

We should comment on these results. (1) It was already known in [25] that the Lee-Yang case relates to the Rogers-Ramanujan identities. We only revisit the same result from another path, the P-partition counting. (2) Note that the two Hilbert series both include the extra factor $(q; q)_\infty$ in the denominators in (7.16), (7.19), compared to the results in [25]. This factor seems to come from the fact that SH^c contains not only the \mathcal{W}_N -algebra but also the Heisenberg (or $U(1)$ current) algebra.

7.4 Conjectures from the general (N, M) cases

As noted before, we cannot obtain the Jordan-Hölder set for the poset X for the general cases which makes the computation of P-partition difficult. Since we have established that the partition for the poset X is equivalent to the Hilbert space of minimal models relevant in the level-rank duality, the partition function for the poset should be equal to the character formula for minimal models [25]:

$$Z_X(q) = \frac{\chi_N(q)}{(q; q)_\infty (q^{N+M}; q^{N+M})_\infty}, \quad (7.20)$$

where, for \mathcal{W}_N minimal models with the level-rank duality, the character $\chi_N(q)$ takes of the form [11]

$$\chi_N(q) = \frac{(q^{N+M}; q^{N+M})_\infty^N \prod_{i < j} (q^{x_i - x_j}; q^{N+M})_\infty \prod_{i > j} (q^{N+M + x_i - x_j}; q^{N+M})_\infty}{(q; q)_\infty^{N-1}}, \quad (7.21)$$

where $x_i = \sum_{j=i}^N n_j$. It is known in [11] that the $\chi_N(q)$ is invariant under replacements $N \mapsto M$ and $x \mapsto y$ (see subsection 2.2) corresponding to the level-rank duality, which gives the character on the \mathcal{W}_M -algebra side.

In summary, we have the following conjecture:

Conjecture. *For the Hilbert series of the SH^c module with the level-rank duality, the following identity holds;*

$$\sum_{\lambda \in \mathcal{A}(X)} q^{|\lambda|} = \frac{(q^{N+M}; q^{N+M})_{\infty}^{N-1} \prod_{i < j} (q^{x_i - x_j}; q^{N+M})_{\infty} \prod_{i > j} (q^{N+M+x_i - x_j}; q^{N+M})_{\infty}}{(q; q)_{\infty}^N}, \quad (7.22)$$

where the partially ordered set X is defined by (6.11) and (7.1), and the $\mathcal{A}(X)$ is the set of all the P -partitions (7.2) over X . ■

8 Constraints on the Hilbert space of general minimal models: N -Burge condition

In this section, we first derive the N -Burge condition, which was obtained in [26–28] for the $N = 2$ case and in [15] from a family of zeros in Λ^+ . We note that similar result was obtained a few years ago in [21] for the q -deformed case. Then we show that this is the sufficient restriction to get the whole submodule without null states in the representation of SH^c corresponding to a general minimal model. As the proof is almost parallel to those in section 6, our explanation will be focused on the points where the generalization is nontrivial.

8.1 N -Burge condition

The N -Burge condition in our convention reads

$$Y_{i,R} - Y_{i+1,R+(n_i-1)} \geq -(n'_i - 1). \quad (8.1)$$

Here, $Y_{i,R}$ denotes the number of boxes in the R -th row of the i -th Young diagram.

In order to derive it from the SH^c , we consider the factors in

$$\Lambda_p^{(k,+)}(\vec{a}, \vec{Y}) = \left(\prod_{q=1}^N \left(\prod_{\ell=1}^{f_q} \frac{a_p - a_q + A_k(Y_p) - B_{\ell}(Y_q) + \xi}{a_p - a_q + A_k(Y_p) - B_{\ell}(Y_q)} \prod_{\ell=1}^{f_q+1} \frac{a_p - a_q + A_k(Y_p) - A_{\ell}(Y_q) - \xi}{a_p - a_q + A_k(Y_p) - A_{\ell}(Y_q)} \right) \right)^{1/2}.$$

We notice that the condition for the numerator in the second product to vanish is $a_p - a_q + A_k(Y_p) - A_{\ell}(Y_q) - \xi = 0$ for some p, q, k and ℓ . Here, we focus on the case $p = q + 1$.

$$\begin{aligned} 0 &= a_{q+1} - a_q + A_k(Y_{q+1}) - A_{\ell}(Y_q) - \xi \\ &= \beta(r_{k-1}^{q+1} - r_{l-1}^q - n'_q + 1) + (s_l^q - s_k^{q+1} + n_q - 1). \end{aligned} \quad (8.2)$$

To make $r_{k-1}^{q+1} = Y_{q+1,R}$ for some R , we have to set $s_k^{q+1} = R + 1$. Then considering the situation with $s_l^q = R' - 1 = R - n_q$, we have

$$Y_{q,R'} - Y_{q+1,R'+(n_q-1)} = -(n'_q - 1). \quad (8.3)$$

The highest weight vector with all empty Young diagrams implies the inequality in (8.1), we thus know (8.1) is the right condition to impose to extract the submodule including the highest weight.

When we go back to the special case of the level-rank duality, the right-hand side of (8.1) vanishes. Those two rows under consideration have respectively the characterizing integers $Na_i + M - (R - 1)N$ and $Na_{i+1} + M - (R + \tilde{n}_i - 1)N$. Their difference is exactly M , that is to say, (8.1) reduces to

$$\lambda(x + M) \geq \lambda(x).$$

We thus see that in this specific case, as we showed before, the N -Burge condition together with the requirement that all states are labeled by Young diagrams, i.e. $\lambda(x + N) \geq \lambda(x)$, determine the spectrum completely. We also expect that a similar discussion to that in section 6 applies to show the N -Burge condition plus the requirement from the Young diagram filter out all the states generated from the highest weight with $D_{1,l}$.

8.2 Sufficiency of the N -Burge condition in SH^c

Let us again introduce the characterizing number $x = pa_i + q - ps \in X$ for the s -th row in the i -th Young diagram. Then the N -Burge condition and the Young diagram requirement can be rewritten to

$$\begin{aligned} \lambda(x - p) &\leq \lambda(x), \\ \lambda(x - q\tilde{n}'_i - (q - p)) &\leq \lambda(x) + \tilde{n}'_i. \end{aligned} \quad (8.4)$$

We define the ordering in X as follows: if there exist positive integers l , and $\exists i < j$, s.t.

$$y = x - lp - (j - i)(q - p) - q \sum_{k=i}^{j-1} \tilde{n}'_k, \quad (8.5)$$

then $x <_X y$. Under this ordering, we can reproduce almost the whole part of the proof we provided in section 6.3. However, in (6.17), we used the fact that the coefficients before N and M when we connect x and y are integers. In more general cases, $\lambda(x) - \lambda(y)$ should also take the form of $\sum \tilde{n}_i$ plus something not independent of this summation, which is quite non-trivial and makes the proof worrying. Albeit, in the following, we show that this problem is naturally resolved in our previous discussion.

Let us focus on $(x - y) + (\lambda(x) - \lambda(y))q$, other factors can be dealt similarly. We can easily see from the definition that X is connected. Therefore we can find some appropriate integers l , i and j to relate x and y as $y = x - lp - (j - i)(q - p) - q \sum_{k=i}^{j-1} \tilde{n}'_k$. When we assume that the factor under consideration vanishes, we have for some integer α ,

$$l - (j - i) = \alpha q, \quad (8.6)$$

$$(j - i) + \sum_{k=i}^{j-1} \tilde{n}'_k = -(\lambda(x) - \lambda(y)) - \alpha p. \quad (8.7)$$

Notice that $\sum_{i=1}^N \tilde{n}'_i = p - N$, we can use another integer $j' = j + \alpha$ to rewrite the second equation as

$$-(\lambda(x) - \lambda(y)) = (j' - i) + \sum_{k=i}^{j'-1} \tilde{n}'_k.$$

Substituting this back to the relation between x and y , we obtain

$$y = x - (q - p)(j' - i) - q \sum_{k=i}^{j'-1} \tilde{n}'_k - p(j' - i),$$

which suggests that if $x \neq y$, we can only have either $x <_X y$ or $x >_X y$. The last two cases can again be denied using the definition of the ordering in X as we have done before because additional terms $\sum \tilde{n}'$ cancel out.

Therefore, the whole discussion in section 6.3 applies again, and we see that the N -Burge condition provides a generalized prescription from that we proposed in section 6. Those two conditions (8.4) here can be treated as a generalized poset construction with which we can count the number of states at each level.

9 Conclusion and discussion

In this paper, we studied some properties of the algebra SH^c . It is supposed to be a universal symmetry which contains the representations of \mathcal{W}_N -algebra for any N . We show that the singular vectors of a \mathcal{W}_N module can be easily understood from the SH^c action on the AFLT basis. Similar computation leads to the N -Burge condition which characterizes the Hilbert space of the minimal models.

The main focus of the paper is to show the triality automorphism in the SH^c algebra which is generated by two mutually noncommuting generators σ_1, σ_2 . While it is the exact automorphism of algebra, it is not straightforward to see the minimal model CFTs are dual to each other since their rank are in general different in the second transformation σ_2 . We have shown an explicit realization of the duality through the minimal models with the level-rank duality. Through a diagrammatic representation of primary fields, we have shown that the infinite number of the central charges of the algebra are kept invariant after nontrivial cancellation of factors. It establishes that the minimal models are connected via the triality of SH^c .

We also examined the structure of the Hilbert space and found that the duality is realized as an explicit automorphism between its N -tuple Young diagram representation and its M -tuple dual. This morphism is given by shuffling Young diagrams row by row (figure 6) and the basis of the module is the set of all the integer partition λ satisfying $\lambda(x - N), \lambda(x - M) \leq \lambda(x)$, (7.2). It is a generalization of the standard partition described by a Young diagram.

Connecting our analysis to the character formulae of minimal models, we have seen that the Hilbert series of P-partitions over a type of partially ordered sets matches the character (7.22). We can prove it explicitly for the Lee-Yang case via the Rogers-Ramanujan identities. Conversely, the equation (7.22) means that, for the Lee-Yang case, the two primary fields give a combinatorial interpretation of the two Rogers-Ramanujan identities, which was already known in [22]. Therefore, this equation may give a general combinatorial interpretation of such identities. Deeper analysis of the corresponding Hasse diagrams is needed to give a complete proof of (7.22).

There are some applications of our results to physics. The immediate one is to 4D/2D correspondence for $\mathcal{N}=2$ supersymmetric gauge theories in 4 dimensions. On the four-dimensional gauge theory side, the factor Λ^\pm in (3.17) is nothing but the ratio of weights of two fixed point in the instanton moduli space. Note that we have obtained the resultant representation by dropping some arrangements of Young diagrams since we have set the Coulomb branch parameter $(a_q)_{q=1}^N$ and the Ω -background parameter $\beta = -\frac{\varepsilon_2}{\varepsilon_1}$ to specific values. This dropping implies that some weights with fixed points vanishes and then, at first sight, such fixed points give divergent contributions to the Nekrasov partition function. However, this divergence comes from the fact that the fixed points with the torus action are not isolated in the moduli space for such a choice of parameters, and then we should not sum all the contributions point by point. Instead, we should integrate over the set of all the fixed points with an appropriate measure. It may be interesting to study the correspondence between the sets of all the fixed points via the level-rank duality, which might imply a duality between instanton moduli spaces with two different gauge groups $SU(N)$ and $SU(M)$ in the special Ω background. In a sense, nonisolated fixed points in the equivariant localization will imply the null states of corresponding minimal models. It may be also interesting to pursue physical meanings of the null states on the 4D side.

The other application is to higher spin gravity theory, where the universal symmetry $\mathcal{W}_\infty[\mu]$ was found. Since both symmetries describe the minimal models of \mathcal{W}_N -algebra, SH^c may have some role there too. In particular, it may be interesting if one can find the role of AFLT basis in the gravitational background. The other possible application is to the fractional quantum Hall effect (FQH). Inspired by a mathematical result [29], it was suggested [30] the Jack polynomial may be interpreted as the ground state wave function. Jack polynomial describes the excited states of the Calogero-Sutherland model and is identified with AFLT basis with $N = 1$ after rewriting the variables by free boson. If Jack polynomial plays some role, so should be general AFLT basis. N -Burge condition may be interpreted as representing a generalized statistics for anyon system.

Acknowledgments

We are grateful to Yuji Tachikawa for helpful discussions and comments. RZ would like to sincerely thank HIROSE international scholarship for generous financial support and K.Goto for inspiring consultation. YM would like to thank Andrei Mironov and Alexei Morozov for inviting him to workshop “Quantum Geometry, Duality and Matrix Models”, at Lebedev Physical Institute and IITP in Moscow (August 24-30, 2015) where part of the content of the paper was presented. Support from JSPS/RFBR bilateral collaboration “Faces of matrix models in quantum field theory and statistical mechanics” which makes this visit possible is also gratefully appreciated. He is obliged to Tomas Prochazka for the correspondence on the triality of SH^c , Mikhail Bernshtein to point out the relevance of [21] and Hiroaki Kanno for the comments on the poset. He is partially supported by Grants-in-Aid for Scientific Research (Kakenhi #25400246).

A Concrete correspondence of states for $(N, M) = (2, 3)$

This appendix gives the detailed correspondence between states labeled by various Young diagrams. As null states have no correspondence in the dual picture, they certainly will be omitted from the following list.

For $\Delta = 0$ case, the set which labels the rows is $X = \{0, -2, -3, -4, \dots\}$. In the following we write $(\lambda(0), \lambda(-2), \lambda(-3), \lambda(-4), \dots)$ to represent the partition. At level one (one state):

$$|\square, \emptyset\rangle \leftrightarrow |\square, \emptyset, \emptyset\rangle \leftrightarrow (1, 0, 0, 0, \dots).$$

At level two (three states),

$$\begin{aligned} |\square, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle \leftrightarrow (1, 0, 1, 0, \dots), \\ |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \emptyset\rangle \leftrightarrow (1, 1, 0, 0, \dots), \\ |\square\square, \emptyset\rangle &\leftrightarrow |\square\square, \emptyset, \emptyset\rangle \leftrightarrow (2, 0, 0, 0, \dots). \end{aligned}$$

At level three (5 states):

$$\begin{aligned} |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square, \emptyset\rangle \leftrightarrow (1, 1, 1, 0, \dots), \\ |\square\square, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle \leftrightarrow (2, 0, 1, 0, \dots), \\ |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \square\rangle \leftrightarrow (1, 1, 0, 1, 0, \dots), \\ |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square\square, \square, \emptyset\rangle \leftrightarrow (2, 1, 0, 0, \dots), \\ |\square\square, \emptyset\rangle &\leftrightarrow |\square\square, \emptyset, \emptyset\rangle \leftrightarrow (3, 0, 0, 0, \dots). \end{aligned}$$

At level four (9 states): (we omit the partition in the following)

$$\begin{aligned} |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \square, \square\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square, \emptyset\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle, \\ |\square\square, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle, & |\square\square, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle, & |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \square\rangle, \\ |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \square, \emptyset\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square\square, \square, \emptyset\rangle, & |\square\square, \emptyset\rangle &\leftrightarrow |\square\square, \emptyset, \emptyset\rangle. \end{aligned}$$

At level five (14 states):

$$\begin{aligned} |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|} \hline \square \\ \hline \end{array}, \square, \square\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square, \square\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \square\rangle, \\ |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square, \emptyset\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square, \emptyset\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle, \\ |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \square, \emptyset\rangle, & |\square\square, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle, & |\square\square, \square\rangle &\leftrightarrow |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset, \emptyset\rangle, \\ |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \square, \square\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \square, \square\rangle, & |\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square, \square, \square, \emptyset\rangle, \\ |\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \emptyset\rangle &\leftrightarrow |\square\square, \square, \emptyset\rangle, & |\square\square, \emptyset\rangle &\leftrightarrow |\square\square, \emptyset, \emptyset\rangle. \end{aligned}$$

For $\Delta = -1/5$, $X = \{-1, -2, -3, -4, \dots\}$. We have two states at the first level:

$$\begin{aligned} |\square, \emptyset\rangle &\leftrightarrow |\emptyset, \square, \emptyset\rangle \leftrightarrow (1, 0, 0, 0, \dots), \\ |\emptyset, \square\rangle &\leftrightarrow |\square, \emptyset, \emptyset\rangle \leftrightarrow (0, 1, 0, 0, \dots). \end{aligned}$$

where $\eta = \pm 1$ and $D_\eta(z) = \sum_{n=0}^{\infty} D_{\eta,n} z^{-n}$. The first relation (B.1) depends on β . One can confirm that it is invariant under triality symmetry:

$$\sigma_1 : f(z, 1/\beta) = f(-\beta z, \beta), \quad (\text{B.3})$$

$$\sigma_2 : f\left(z, \frac{\beta}{\beta-1}\right) = f((\beta-1)z, \beta). \quad (\text{B.4})$$

With these properties, (B.1) becomes invariant under transformations $\sigma_1 : D'_{\pm 1, k} = \gamma^{\pm 1}(-\beta)^{-k} D_{\pm 1, k}$, $\sigma_2 : D'_{\pm 1, \ell} = \gamma^{\pm 1}(\beta-1)^{1/2-\ell} D_{\pm 1, \ell}$, which were derived in section 3.2.

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